Mathematical Modelling of System Lifetimes under Cascading Effect and Dependence

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Abstract: - In this paper, we discuss the lifetimes of systems composed of components having interdependence and cascading effect. For this, a class of bivariate distributions is constructed based on the notion of conditional failure rate. We show that the members in a suitably defined subclass of the developed bivariate distributions satisfy the bivariate lack of memory property. Based on the developed class of distributions, we study the lifetimes of systems having interdependence and cascading effect. We compare the lifetimes of systems having interdependence and cascading effect with those of systems having independent components. The obtained results can be usefully applied to civil engineering.

Key-Words: -Mathematical modelling of lifetimes, Cascading effect, Dependence, Load-sharing components, Parallel system, Bivariate distribution, Civil engineering

1 Introduction

Dependent random quantities can frequently be encountered in practice and they have been modelled by using bivariate distributions. In the literature, various specific parametric models for bivariate distributions have been suggested and studied. See, for instance, Freund (1961), Marshall and Olkin (1967), Downton (1970), Hawkes (1972), Block and Basu (1974) and Shaked (1984). A nice review on the modelling of multivariate survival models can be found in Hougaard (1987). An excellent encyclopaedic survey of various bivariate distributions can be found in Balakrishnan and Lai (2009).

In this paper, taking into account the physics of failure of items and the interrelationship between them, we propose and discuss a general methodology for constructing a 'general class of bivariate distributions'. Based on the developed class, we study the lifetimes of systems having interdependence and cascading effect.

The structure of this paper is as follows. In Section 2, a general methodology for constructing a new class of bivariate distributions is suggested. It is shown that a suitably defined subclass of the developed bivariate distributions satisfy the bivariate lack of memory property. In Section 3, we study the lifetimes of systems having interdependence and cascading effect based on the developed bivariate distribution. Finally, in Section 4, the results in this paper are briefly summarized. Furthermore, some topics for the future study are suggested and concluding remarks are given.

2 Modelling of Bivariate Distributions

In order to describe the stochastic dependence discussed in this paper, we need the concept of 'conditional failure rate' or 'failure (hazard) rate process' (or random failure rate). See, e.g., Kebir (1991), Aven and Jensen (1999) and Finkelstein and

Cha (2013) for the definition and relevant discussions of this concept.

Suppose that the system is composed of two components: component 1 and component 2 and they start to operate at time $t=0$. The original lifetimes of components 1 and 2, when they start to operate, are described by the corresponding failure rates $\lambda_1(t)$ and $\lambda_2(t)$, respectively. These original lifetimes of components 1 and 2 are denoted by $X_1{}^*$ and $X_2{}^*$, respectively, assuming that $X_1{}^*$ and $X_2{}^*$ are stochastically independent.

Similar to Freund (1961)'s model, we consider the practical situation when the failure of one component increases the stress of the other component, which results in the shortened residual lifetime of the remaining component. Thus, there is a change point, $\min\{X_1{}^*, X_2{}^*\}$, after which the residual lifetime distribution of the remaining component changes. Under this type of dependency, we denote the corresponding eventual lifetimes of components 1 and 2 by X_1 and X_2, respectively.

Define $\Psi_1(t)=1$ ($\Psi_2(t)=1$) if component 1 (component 2) is functioning at time t, whereas $\Psi_1(t)=0$ ($\Psi_2(t)=0$) if component 1 (component 2) is at failed state at time t. For notational convenience, let $\tilde{i}=2$ when $i=1$; whereas $\tilde{i}=1$ when $i=2$. Then, we assume the conditional failure rate of component i is given by

$$\begin{aligned}&r_i(t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s \le t)\\ &\equiv \lim_{\Delta t \to 0} P(t < X_i \le t+\Delta t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s \le t, X_i > t)\\ &= \lambda_i(t),\ i=1,2,\end{aligned} \tag{1}$$

and

$$\begin{aligned}&r_i(t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u; \Psi_{\tilde{i}}(s)=0, u \le s \le t)\\ &\equiv \lim_{\Delta t \to 0} P(t < X_i \le t+\Delta t \mid\\ &\qquad \Psi_{\tilde{i}}(s)=1, 0 \le s < u; \Psi_{\tilde{i}}(s)=0, u \le s \le t, X_i > t)\\ &= \alpha_i(t-u)\lambda_i(t),\ t \ge u,\ i=1,2,\end{aligned} \tag{2}$$

where $\alpha_i(w) \ge 1$, for all $w \ge 0$, $i=1,2$. Note that the effect of the elapsed time from the change point to the point of interest can be modeled by the argument "w". Obviously, the cases of $\alpha_i(w) \equiv 1$, for all $w \ge 0$, $i=1,2$, in Equation (2) correspond to the independence of X_1 and X_2.

Let $\tilde{X}_i$, $i=1,2$, denote the lifetimes of component 1 and component 2 when the two lifetimes are completely independent, i.e., when $\alpha_i(w) \equiv 1$, $i=1,2$. Then,

$$\begin{aligned}&P(\tilde{X}_i > t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u,\ \Psi_{\tilde{i}}(s)=0, s \ge u, \tilde{X}_i > u)\\ &\qquad = \exp\left(-\int_0^{t-u} \lambda_i(u+w)\,dw\right),\ i=1,2,\end{aligned}$$

whereas, when the two lifetimes X_1 and X_2 are dependent in accordance with the above described model,

$$\begin{aligned}&P(X_i > t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u, \Psi_{\tilde{i}}(s)=0, s \ge u, X_i > u)\\ &\qquad = \exp\left(-\int_0^{t-u} \alpha_i(w)\lambda_i(u+w)\,dw\right), i=1,2.\end{aligned}$$

In the following discussions, the notations $S(x_1, x_2)$, $f(x_1, x_2)$ will be used to denote the joint survival function and the joint pdf of X_1 and X_2, respectively. We will now obtain the joint distribution of X_1 and X_2 under the assumed model. Observe that

$$\begin{aligned}&P(X_1 > x_1, X_2 > x_2)\\ &= P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* > X_2{}^*)P(X_1{}^* > X_2{}^*)\\ &\quad + P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* < X_2{}^*)P(X_1{}^* < X_2{}^*).\end{aligned} \tag{3}$$

Here,

$$\begin{aligned}&P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* < X_2{}^*)\\ &= \int_0^\infty P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* < X_2{}^*, X_1{}^* = u)\\ &\qquad \times f_{X_1{}^* \mid X_1{}^* < X_2{}^*}(u)\,du,\end{aligned} \tag{4}$$

$$\begin{aligned}&P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* > X_2{}^*)\\ &= \int_0^\infty P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* > X_2{}^*, X_2{}^* = u)\\ &\qquad \times f_{X_2{}^* \mid X_1{}^* > X_2{}^*}(u)\,du.\end{aligned} \tag{5}$$

(Case I) Let $0 < x_1 < x_2$. The conditional joint survival function

$$P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* < X_2{}^*, X_1{}^* = u)$$

is, for $u < x_1$ or $x_2 \le u$, obviously given by 0 or 1, respectively. For $x_1 \le u < x_2$, from the assumptions on the conditional failure rates stated in (1)-(2), we have

$$\begin{aligned}&P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* < X_2{}^*, X_1{}^* = u)\\ &= \exp\left(-\int_0^{x_2-u} \alpha_2(w)\lambda_2(u+w)\,dw\right),\ x_1 \le u < x_2.\end{aligned}$$

On the other hand, the conditional joint survival function $P(X_1 > x_1, X_2 > x_2 \mid X_1{}^* > X_2{}^*, X_2{}^* = u)$ is given by 0 if $u < x_2$; and is given by 1 if $u \ge x_2$.

Now combining (3)-(5) based on the above derivations, we have the joint survival function for $0 < x_1 < x_2$.

(Case II) Let $0 < x_2 \le x_1$. As the roles of (X_1^*, X_1) and (X_2^*, X_2) are symmetric, the result can be obtained symmetrically.

The joint pdf $f(x_1, x_2)$ can be obtained by differentiating $S(x_1, x_2)$. See Lee and Cha (2014) for a more detailed derivation under a more general setting. We thus now have the following result.

Proposition 1. Under the conditional failure rate model stated in (1)-(2), the joint survival function $S(x_1, x_2)$, for $0 < x_1 < x_2$, is given by

$$S(x_1, x_2) = \int_{x_1}^{x_2} \lambda_1(u) \exp\left(-\int_0^{x_2-u} \alpha_2(w)\lambda_2(u+w)dw \right) \times \exp\left(-\int_0^u \lambda_1(w) + \lambda_2(w)\,dw \right) du + \exp\left(-\int_0^{x_2} \lambda_1(w) + \lambda_2(w)\,dw \right), \quad \text{for} \quad 0 < x_1 < x_2, \tag{6}$$

and $S(x_1, x_2)$, for $0 < x_2 \le x_1$, can be obtained symmetrically by replacing x_1, x_2, $\lambda_1(\cdot)$, $\lambda_2(\cdot)$, $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ in the right-hand side of Equation (3) with respective opposite components. The corresponding joint pdf, for $0 < x_1 < x_2$, is given by

$$f(x_1, x_2) = \lambda_1(x_1)\lambda_2(x_2)\alpha_2(x_2 - x_1) \times \exp\left(-\int_0^{x_1} \lambda_1(w) + \lambda_2(w)\,dw \right) \exp\left(-\int_0^{x_2-x_1} \alpha_2(w)\lambda_2(x_1+w)\,dw \right), \text{for } 0 < x_1 < x_2,$$

and $f(x_1, x_2)$, for $0 < x_2 \le x_1$, can also be obtained symmetrically.

By letting $x_1 \equiv 0$ or $x_2 \equiv 0$ in $S(x_1, x_2)$, the marginal distributions can be obtained. See Lee and Cha (2014) for more detailed discussions of the properties of the developed class.
The most typical practical example to which the developed bivariate distributions can be applied is as follows.

Example 1. (Load-Sharing Components)
In a load-sharing system, when any one of the components fails, its load is automatically transmitted to the remaining component. This results in a higher load on the surviving component, thereby inducing a higher failure rate for it. This introduces failure dependency among the load sharing components. Examples of load-sharing components include electric generators sharing an electrical load in a power plant, CPUs in a multiprocessor computer system, cables in a suspension bridge, and valves or pumps in a hydraulic system (see Amari and Bergman (2008)).

A well-defined subclass of the general class defined in Proposition 1 shares the 'bivariate lack of memory property (BLMP)' with the Freund's bivariate exponential distribution. It has been known that Freund's bivariate exponential distribution is 'one of the few absolutely continuous bivariate distributions' which possess the BLMP. The definition of BLMP is as follows.

Definition 1. (Bivariate Lack of Memory Property). A bivariate random variable (X_1, X_2) is said to have the bivariate lack of memory property (BLMP) iff

$$P(X_1 > t + s_1, X_2 > t + s_2 \mid X_1 > t, X_2 > t) = P(X_1 > s_1, X_2 > s_2), \text{ for all } t, s_1, s_2 \ge 0. \tag{7}$$

Interpreting (7), if both of the two items are alive at t, then the joint distribution of their remaining lifetimes is the original joint distribution.
Suppose that $\lambda_i(t) = \lambda_i,\, t \ge 0$, $i = 1,2$, in (6). For $s_2 > s_1$, from Proposition 1,

$$P(X_1 > t + s_1, X_2 > t + s_2) = \int_{t+s_1}^{t+s_2} \lambda_1 \exp\left(-\lambda_2 \int_0^{t+s_2-u} \alpha_2(w)dw \right) \exp\{-(\lambda_1 + \lambda_2)u\} du + \exp\{-(\lambda_1 + \lambda_2)(t + s_2)\}.$$

By changing $u - t = v$ in the integral, we can show the equality in (7). For the case $s_1 \ge s_2$, the result can be shown symmetrically. See Lee and Cha (2014) for a more detailed proof. We thus have now the following proposition, which states that all the members belonging to a well-defined subclass possess BLMP.

Proposition 2. Suppose that $\lambda_i(t) = \lambda_i,\, t \ge 0$, $i = 1,2$. Then (X_1, X_2) possesses the BLMP.

3 System Lifetimes

In this section, we study the lifetimes of systems when the baseline distributions are Weibull. Let $\lambda_i(t) = \mu_i \gamma_i (\mu_i t)^{\gamma_i - 1}, t \ge 0$, $i = 1,2$, and $\alpha_i(t) = \alpha_i t + 1$, $\alpha_i > 0$, $i = 1,2$. In this case, from

Proposition 1, the joint survival function of (X_1, X_2) is given by

$$S(x_1,x_2) = \exp\left(-\frac{\gamma_2 \alpha_2 \mu_2^{\gamma_2}}{\gamma_2 + 1} x_2^{\gamma_2+1} - \mu_2^{\gamma_2} x_2^{\gamma_2} \right)$$
$$\times \int_{x_1}^{x_2} \mu_1^{\gamma_1} \gamma_1 u^{\gamma_1 - 1} \exp\left(\alpha_2 \mu_2^{\gamma_2} x_2^{\gamma_2} u - \frac{\alpha_2 \mu_2^{\gamma_2}}{\gamma_2 + 1} u^{\gamma_2+1} - \mu_1^{\gamma_1} u^{\gamma_1} \right) du$$
$$+ \exp\left(-\mu_1^{\gamma_1} x_2^{\gamma_1} - \mu_2^{\gamma_2} x_2^{\gamma_2} \right), \quad \text{for } 0 < x < y$$

,

$$S(x_1,x_2) = \exp\left(-\frac{\gamma_1 \alpha_1 \mu_1^{\gamma_1}}{\gamma_1 + 1} x_1^{\gamma_1+1} - \mu_1^{\gamma_1} x_1^{\gamma_1} \right)$$
$$\times \int_{x_1}^{x_2} \mu_2^{\gamma_2} \gamma_2 u^{\gamma_2 - 1} \exp\left(\alpha_1 \mu_1^{\gamma_1} x_1^{\gamma_1} u - \frac{\alpha_1 \mu_1^{\gamma_1}}{\gamma_1 + 1} u^{\gamma_1+1} - \mu_2^{\gamma_2} u^{\gamma_2} \right) du$$
$$+ \exp\left(-\mu_1^{\gamma_1} x_1^{\gamma_1} - \mu_2^{\gamma_2} x_1^{\gamma_2} \right), \quad \text{for } 0 < y \le x;$$

and the corresponding joint pdf is given by

$$f(x_1,x_2)$$
$$= \mu_1^{\gamma_1} \gamma_1 x_1^{\gamma_1 - 1} \mu_2^{\gamma_2} \gamma_2 x_2^{\gamma_2 - 1} \{\alpha_2 (x_2 - x_1) + 1\}$$
$$\times \exp\Bigg(-\alpha_2 \mu_2^{\gamma_2} x_2^{\gamma_2} (x_2 - x_1) + \frac{\alpha_2 \mu_2^{\gamma_2}}{\gamma_2 + 1} (x_2^{\gamma_2+1} - x_1^{\gamma_2+1}) - \mu_1^{\gamma_1} x_1^{\gamma_1} - \mu_2^{\gamma_2} x_2^{\gamma_2} \Bigg)$$

, for $0 < x < y$,

$$f(x_1,x_2)$$
$$= \mu_1^{\gamma_1} \gamma_1 x_1^{\gamma_1 - 1} \mu_2^{\gamma_2} \gamma_2 x_2^{\gamma_2 - 1} \{\alpha_1 (x_1 - x_2) + 1\}$$
$$\times \exp\Bigg(-\alpha_1 \mu_1^{\gamma_1} x_1^{\gamma_1} (x_1 - x_2) + \frac{\alpha_1 \mu_1^{\gamma_1}}{\gamma_1 + 1} (x_1^{\gamma_1+1} - x_2^{\gamma_1+1}) - \mu_1^{\gamma_1} x_1^{\gamma_1} - \mu_2^{\gamma_2} x_2^{\gamma_2} \Bigg)$$

, for $0 < y \le x$.

Suppose that the system is a series system. Then the lifetime of the system is $T_S = \min\{X_1, X_2\} = \min\{X_1^*, X_2^*\} = \min\{\tilde{X}_1, \tilde{X}_1\}$. Thus, in this case, the lifetime of a system having interdependence components is the same as that of a system having independent components. We now assume that the system is a parallel system. In this case, the lifetime of a system having interdependence components is $T_S = \max\{X_1, X_2\}$. In order to obtain the distribution of T_S, define $T_1 = \min\{X_1, X_2\}$ and $T_2 = \max\{X_1, X_2\}$. Then the joint pdf of (T_1, T_2), $g(t_1, t_2)$, is given by

$$g(t_1,t_2) = f(t_1,t_2) + f(t_2,t_1), \ t_1 \le t_2,$$

and thus, the pdf of T_S is given by

$$f_{T_S}(t) = \int_0^t g(t_1, t) dt_1, \ t \ge 0.$$

Cleary, the pdf of lifetime of a system having independent components can be obtained by setting $\alpha_i = 0$, $i = 1,2$. The graphs of the survival function of the system having interdependence components when $\mu_i = 0.5$, $\gamma_i = 3$, $\alpha_i = 2$, $i = 1,2$, and that of the system having independent components are given in Figure 1.

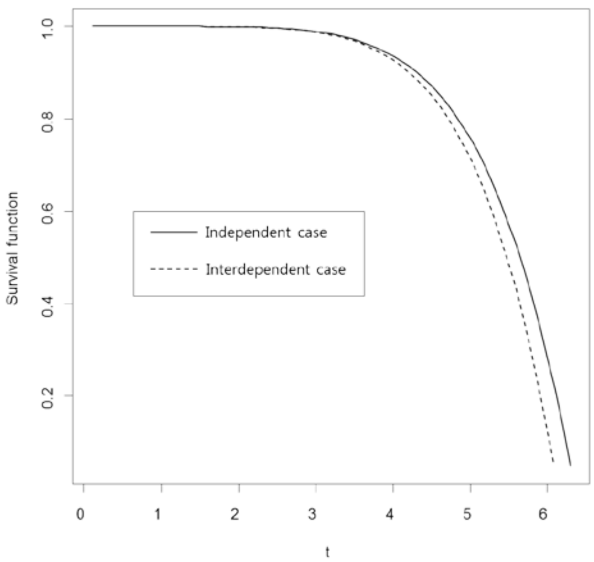

Fig. 1. The survival functions

4 Conclusion

In this paper, our discussions are mainly focused on generating bivariate models. However, similar approach could be applied to generate a new class of multivariate distributions. For example, suppose that we have three components. When we had two components, depending on the states of the other component, two kinds of conditional failure rates for each component were needed for the construction (Equations (1) and (2)). When we have three components, depending on the states of the other two components, five kinds of conditional failure rates for each component are needed for the construction of a class of trivariate distributions. For example, for component 1, the following five kinds of conditional failure rates should be defined:

$r_1(t \mid \Psi_2(s) = 1, 0 \le s \le t; \Psi_3(s) = 1, 0 \le s \le t)$;

$r_1(t \mid \Psi_2(s) = 1, 0 \le s < u; \Psi_2(s) = 0, u \le s \le t;$
$\Psi_3(s) = 1, 0 \le s \le t)$;

$r_1(t \mid \Psi_3(s) = 1, 0 \le s < u; \Psi_3(s) = 0, u \le s \le t;$
$\Psi_2(s) = 1, 0 \le s \le t)$;

$r_1(t \mid \Psi_2(s) = 1, 0 \le s < u_1; \Psi_2(s) = 0, u_1 \le s \le t;$

$$\Psi_3(s) = 1, 0 \le s < u_2; \Psi_3(s) = 0, u_2 \le s \le t)$$

;

$r_1(t \mid \Psi_3(s) = 1, 0 \le s < u_1; \Psi_3(s) = 0, u_1 \le s \le t;$

$$\Psi_2(s) = 1, 0 \le s < u_2; \Psi_2(s) = 0, u_2 \le s \le t),$$

where $u_1 \le u_2$ and $\Psi_i(t)$ represents the state of component i, $i = 2,3$. Then, applying approach and mechanism similar to those given in the proof of Proposition 1, a new class of trivariate distributions could be constructed.

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