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# Some new type of difference sequence space of nonabsolute type and some matrix transformation

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Abstract: In this paper, we introduce the space $r^q(\triangle_u^p)$, where

$$r^q(\triangle_u^p) = \{x = (x_k) \in \omega : (u_k \triangle x_j) \in r^q(u,p) < \infty\};$$

where $r^q(u,p)$ has recently been studied by Neyaz and Hamid. We show its completeness property, prove that the space $r^q(\triangle_u^p)$ and $l(p)$ are linearly isomorphic and compute their $\alpha$ -, $\beta$- and $\gamma$-duals. Furthermore construct the basis of $r^q(u,p)$. In our last section we characterize some matrix class.

Key–Words: Sequence space of non-absolute type; paranormed sequence space; $\alpha$-, $\beta$- and $\gamma$-duals ; matrix transformations.

## 1 Introduction

We denote the set of all sequences with complex terms by $\omega$. It is a routine verification that $\omega$ is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequencdes which are defined, as usual, by

$$x + y = (x_k) + (y_k) = (x_k + y_k)$$

and

$$\alpha x = \alpha(x_k) = (\alpha x_k),$$

respectively; where $x = (x_k)$, $y = (y_k) \in \omega$ and $\alpha \in \mathbf{C}$. By *sequence space* we understand a linear subspace of $\omega$ i.e. the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise adition and scalar multiplication. Throughout the paper $N$, $R$ and $C$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let $l_\infty$, $c$ and $c_0$, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by $l_1$ , $l(p)$, $cs$ and $bs$ we deonte the spaces of all absolutely , $p$-absolutely convergent, convergent and bounded series, respectively.

Let $X, Y$ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in N$. Then, the matrix $A$ defines the $A$-transformation from $X$ *into* $Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$ exists and is in $Y$; where $(Ax)_n = \sum_k a_{nk} x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $A \in (X : Y)$ we mean the characterizations of matrices from $X$ to $Y$ *i.e.*, $A : X \to Y$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$ which is called as the $A$-limit of $x$.

For a sequence space $X$, the matrix domain $X_A$ of an infinite matrix $A$ is defined as

$$X_A = \{x = (x_k) : Ax \in X\}. \tag{1}$$

The theory of matrix transformations is a wide field in summability; it deals with the characterisations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [26] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let $A = (a_{nk})$ be any matrix. Then a sequence $x$ is said to be summable to $l$, written $x_k \to l$, if and only if $A_n x = \sum\limits_k a_{nk}x_k$ exists for each $n$ and $A_n x \to l$ $(n \to \infty)$. For example, if $I$ is the unit matrix, then $x_k \to l(I)$ means precisely that $x_k \to l$ $(k \to \infty)$, in the ordinary sense of convergence.

We denote by $(A)$ the set of all sequences which are summable $A$. The set $(A)$ is called *summability field* of the matrix $A$. Thus, if $Ax = (a_n(x))$, then $(A) = \{x \ : \ Ax \in c\}$, where $c$ is the set of convergent sequences. For example, $(I) = c$.

A infinite matrix $A = (a_{nk})$ is said to be regular [15] if and only if the following conditions (or Toplitz conditions) hold:

$$(i) \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

$$(ii) \quad \lim_{n\to\infty} a_{nk} = 0, \quad (k = 0, 1, 2, ...),$$

$$(iii) \quad \sum_{k=0}^{\infty} |a_{nk}| < M, \quad (M > 0, \ n = 0, 1, 2, ...).$$

Let $(q_k)$ be a sequence of positive numbers and let us write,$Q_n = \sum_{k=0}^{n} q_k$ for $n \in N$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean $(R, q_n)$ is given by

$$r_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \le k \le n, \\ \\ 0 & \text{if } k > n \end{cases}$$

The Riesz mean $(R, q_n)$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$ ( see, Petersen [22, p.10 ] ).

Kizmaz [11] defined the difference sequence spaces $Z(\triangle)$ as follows

$$Z(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in Z\}$$

where, $Z \in \{l_\infty, c, c_0\}$ and $\triangle x_k = x_k - x_{k+1}$.

Bas̆ar and Altay[2] has studied the sequence space as

$$bv_p = \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty \right\},$$

where $1 \le p < \infty$. With the notation of (1), the space $bv_p$ can be redefined as

$$bv_p = (l_p)_\triangle, 1 \le p < \infty$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n - 1 \le k \le n, \\ \\ 0 & \text{if } k < n - 1 \text{ or } k > n. \end{cases}$$

The approach of constructing a new sequence space by means of matrix domain of a particular limitation mehtod has been studied by several authors. They introduced the sequence spaces $(l_\infty)_{N_q}$ and $c_{N_q}$(see, [27]), $(l_p)_{C_1} = X_p$ and $(l_\infty)_{C_1} = X_\infty$(see, [21]), $(l_\infty)_{R^t} = r^t_\infty$, $(c)_{R^t} = r^t_c$ and $(c_o)_{R^t} = r^t_0$(see, [16]), $(l_p)_{R^t} = r^t_p$(see, [1]), $(c_0)_{E^r} = e^r_0$ and $(c)_{E^r} = e^r_c$(see, [2]), $(l_p)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$(see, [3), $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$(see, [4]), $[c_0(u,p)]_{A^r} = a^r_0(u,p)$ and $[c(u,p)]_{A^r} = a^r_c(u,p)$(see, [5], $(l_p)_{A^r} = a^r_p$ and $(l_\infty)_{A^r} = a^r_\infty$(see, [6], $(c_0)_{C_1} = \widehat{c}_0$, $c_{C_1} = \widehat{c}$(see, [23], $c^\lambda_0(\triangle) = \left(c^\lambda_0\right)_\triangle$ and $c^\lambda(\triangle) = \left(c^\lambda\right)_\triangle$(see, [20], $\mu_G = Z(u,v,\mu)$(see, [17), Neyaz and Hamid $r^q(u,p) = \{l(p)\}_{R^q}$(see, [24]); where $N_q, C_1, R^t$ and $E^r$ denotes the Nörland, Cesäro, Riesz and Eular means, respectively, $A^r$ and $C$ are respectively defined in [17, 19], $\mu = \{c_0, c, l_p\}$ and $1 \le p < \infty$.

2. The Riesz Sequence space $r^q(\triangle^p_u)$ of non-absolute type :

In this section, we define the Riesz sequence space $r^q(\triangle^p_u)$ , and prove that the space $r^q(\triangle^p_u)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$.

A linear Topological space $X$ over the field of real numbers $R$ is said to be a paranormed space if there is a subadditive function $h : X \to R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and

$h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha's$ in $R$ and $x's$ in $X$ , where $\theta$ is a zero vector in the linear space $X$. Assume here and after that $(p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$and $M = max\{1, H\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [11] (see also, [25, 28-29]) as follows :

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

and

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which are complete spaces paranormed by

$h_1(x) = [\sum_k |x_k|^{p_k}]^{1/M}$ and $h_2(x) = \sup_k |x_k|^{p_k/M}$

iff $\inf p_k > 0$.

We shell assume throughout that $p_k^{-1} + \{p_k^{'}\}^{-1}$ provided $1 < inf p_k \leq H < \infty$ and we denote the collection of all finite subsets of $N$ by $F$, where $N$={0, 1, 2, ... }.

Following Basar and Altay [2], Basar, Altay and Mursaleen [3], Choudhary and Mishra [7], Hamid [8], Hamid and Neyaz[9], Mursaleen [18], Mursaleen, Basar and Altay [19], Neyaz and Hamid[24], we define the Reisz sequence space $r^q(\triangle_u^p)$ as the set of all sequences such that $R^q$ transform of it is in the space $l(p)$, that is,

$$r^q(\triangle_u^p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_k q_j \triangle x_j \right|^{p_k} < \infty \right\}$$

where, $0 < p_k \leq H < \infty$.

Remark 2.0 : Incase $\triangle x_k = x_k$ (fixed) for all $k \in N$, the sequence spaces $r^q(\triangle_u^p)$ reduces to $r^q(u,p)$, introduced by Neyaz and Hamid [24]. Also, if $(u_k) = e = (1, 1, ...)$ and $\triangle x_k = x_k$ (fixed) for all $k \in N$, the sequence spaces $r^q(\triangle_u^p)$ reduces to $r^q(p)$, introduced by Altay and Basar [1]. Further, for $(u_k) = e = (1, 1, ...)$, $\triangle x_k = x_k$ for all $k \in N$, and $q_n = 1$ for all $n \in N$, the sequence spaces $r^q(\triangle_u^p)$ reduces to space of Arithmetic means $X_p$ of nonabsolute type introduced and studied by Ng and Lee [21].

With the notation of (1) that

$$r^q(\triangle_u^p) = \{l(p)\}_{R_u^q}.$$

Define the sequence $y = (y_k)$, which will be used, by the $R^q$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} u_k q_j \triangle x_j. \tag{2}$$

Now, we begin with the following theorem which is essential in the text.

Theorem 2.1 : $r^q(\triangle_u^p)$ is a complete linear metric space paranormed by $h_\triangle$, defined as

$$h_\triangle(x) = \left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_j - q_{j+1}) x_j + \frac{q_k u_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}}$$

with $0 < p_k \leq H < \infty$.

Proof: The linearity of $r^q(\triangle_u^p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from from the inequalities which are satisfied for $z, x \in r^q(\triangle_u^p)$( see [12], p.30] )

$$\left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_j - q_{j+1})(x_j + z_j) + \frac{q_k u_k}{Q_k}(x_k + z_k) \right|^{p_k} \right]^{\frac{1}{M}}$$
$$\leq \left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_j - q_{j+1}) x_j + \frac{q_k u_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}}$$
$$+ \left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k(q_j - q_{j+1}) z_j + \frac{q_k u_k}{Q_k} z_k \right|^{p_k} \right]^{\frac{1}{M}} \tag{3}$$

and for any $\alpha \in \mathbf{R}$ (see, [14])

$$|\alpha|^{p_k} \leq max(1, |\alpha|^M). \tag{4}$$

It is clear that, $h_\triangle(\theta)$=0 and $h_\triangle(x) = h_\triangle(-x)$ for all $x \in r^q(\triangle_u^p)$. Again the inequality (3) and (4), yield the subadditivity of $h_\triangle$ and

$$h_\triangle(\alpha x) \leq max(1, |\alpha|) h_\triangle(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(\triangle_u^p)$ such that $h_\triangle(x^n - x) \to 0$ and $(\alpha_n)$ is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$h_\triangle(x^n) \leq h_\triangle(x) + h_\triangle(x^n - x)$$

holds by subadditivity of $h_\triangle$ ,$\{h_\triangle(x^n)\}$ is bounded and we thus have

$$h_\triangle(\alpha_n x^n - \alpha x) =$$

$$\left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_k(q_j - q_{j+1})(\alpha_n x_j^n - \alpha x_j) \right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq |\alpha_n - \alpha|^{\frac{1}{M}} h_\triangle(x^n) + |\alpha|^{\frac{1}{M}} h_\triangle(x^n - x)$$

which tends to zero as $n \to \infty$ . That is to say that the scalar multiplication is continuous. Hence, $h_\triangle$ is paranorm on the space $r^q(\triangle_u^p)$.

It remains to prove the completeness of the space $r^q(\triangle_u^p)$. Let $\{x^j\}$ be any Cauchy sequence in the space $r^q(\triangle_u^p)$ , where $x^i = \{x_0^i, x_1^i, ...\}$. Then , for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$h_\triangle(x^i - x^j) < \epsilon \tag{5}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $h_\triangle$ and for each fixed $k \in N$ that

$$\left|(R^q \triangle x^i)_k - (R^q \triangle x^j)_k\right|$$

$$\leq \left[\sum_k \left|(R^q \triangle x^i)_k - (R^q \triangle x^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R^q \triangle x^0)_k, (R^q \triangle x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in N$. Since $R$ is complete, it converges,say, $(R^q \triangle x^i)_k \to ((R^q \triangle x)_k$ as $i \to \infty$. Using these infinitely many limits $(R^q \triangle x)_0, (R^q \triangle x)_1, \ldots$ , we define the sequence $\{(R^q \triangle x)_0, (R^q \triangle x)_1, \ldots\}$.

From (5) for each $m \in N$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{m} \left|(R^q \triangle x^i)_k - (R^q \triangle x^j)_k\right|^{p_k} \leq h_\triangle(x^i - x^j)^M$$

$$< \epsilon^M. \tag{6}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (6) and then $m \to \infty$ , we obtain

$$h_\triangle(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (6) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in N$ that
$[\sum_{k=0}^{m} |(R^q x)_k|^{p_k}]^{\frac{1}{M}}$

$$\leq h_\triangle(x^i - x) + h_\triangle(x^i) \leq 1 + h_\triangle(x^i)$$

which implies that $x \in r^q(\triangle_u^p)$. Since $h_\triangle(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $r^q(\triangle_u^p)$ is complete, hence the proof .

Note that one can easily see the absolute property does not hold on the spaces $r^q(\triangle_u^p)$ , that is $h_\triangle(x) \neq h_\triangle(|x|)$ for atleast one sequence in the space $r^q(\triangle_u^p)$ and this says that $r^q(\triangle_u^p)$ is a sequence space of non-absolute type.

Theorem 2.2 : The Riesz sequence space $r^q(\triangle_u^p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.

Proof : To prove the theorem, we should show the existence of a linear bijection between the spaces $r^q(\triangle_u^p)$ and $l(p)$ , where $0 < p_k \leq H < \infty$. With the notation of (3), define the transformation T from $r^q(\triangle_u^p)$ to $l(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence $T$ is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k$ by

$$x_k = \sum_{n=0}^{k-1} \left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) u_k^{-1} Q_k y_k + u_k^{-1} \frac{Q_k}{q_k} y_k,$$

for $k \in N$. Then,

$$h_{\triangle}(x) = \left[ \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j - q_{j+1}) x_j + \frac{q_k u_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}}$$

$$= \left[ \sum_k \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^{p_k} \right]^{\frac{1}{M}}$$

$$= \left[ \sum_k |y_k|^{p_k} \right]^{\frac{1}{M}}$$

$$= h_1(y) < \infty,$$

where,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j \end{cases}$$

Thus, we have $x \in r^q(\triangle_u^p)$. Consequently, T is surjective and is paranorm preserving. Hence, $T$ is a linear bijection and this says us that the spaces $r^q(\triangle_u^p)$ and $l(p)$ are linearly isomorphic, hence the proof.

3. Basis and $\alpha$-, $\beta$- and $\gamma$-duals of the space $r^q(\triangle_u^p)$ :

In this section, we compute $\alpha$ , $\beta$- and $\gamma$- duals of the space $r^q(\triangle_u^p)$ and finally we give the basis for the space $r^q(\triangle_u^p)$.

For the sequence space $X$ and $Y$, define the set

$$S(X:Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}. \quad (7)$$

With the notation of (7), the $\alpha$-, $\beta$- and $\gamma$-duals of a sequence space X, which are respectively denoted by $X^\alpha$ and $X^\beta$ and are defined by

$X^\alpha = S(X : l_1)$ , $X^\beta = S(X : cs)$ and $X^\gamma = S(X : bs)$.

If a sequence space $X$ paranormed by $h$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)$ such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0$$

then $(b_n)$ is called a Schauder basis (or briefly basis ) for $X$. The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$ and written as $x = \sum \alpha_k b_k$.

First we first state some lemmas which are needed in proving our theorems .

Lemma 3.1 [10, Theorem 5.10] :
(i) Let $1 < p_k \leq H < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \leq 1$.Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{K \in F} \sup_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p_k} < \infty.$$

Lemma 3.2 [12, Theorem 1] :
(i) Let $1 < p_k \leq H < \infty$.Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_n \sum_k |a_{nk} B^{-1}|^{p_k'} < \infty. \quad (8)$$

(ii) Let $0 < p_k \leq 1$ for every $k \in N$ . Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \quad (9)$$

Lemma 3.3 [12, Theorem 1 ] : Let $0 < p_k \leq H < \infty$ for every $k \in N$. Then $A \in (l(p) : c)$ if and only if (8) and (9) hold along with

$$\lim_n a_{nk} = \beta_k \ for \ k \in N \quad (10)$$

also holds.

Theorem 3.4 : Let Let $1 < p_k \leq H < \infty$ for every $k \in N$. Define the sets $D_1(u,p)$ and $D_2(u,p)$ as follows

$$D_1(u,p) = \bigcup_{B>1} \{a = (a_k) \in \omega :$$

$$\sup_{K \in F} \sum_k | \sum_{n \in K} \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_k^{-1} a_n Q_k + \frac{a_n}{q_n} u_k^{-1} Q_n B^{-1} |^{p_k'} < \infty \}$$

and

$$D_2(u,p) = \bigcup_{B>1} \{a = (a_k) \in \omega :$$

$$\sum_k \left| [ \left( \frac{a_k}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k ] B^{-1} \right|^{p_k'} < \infty \}.$$

Then,

$$[r^q(\triangle_u^p)]^\alpha = D_1(u,p)$$

and

$$[r^q(\triangle_u^p)]^\beta = D_2(u,p) \cap \ cs.$$

Proof : Let us take any $a = (a_k) \in \omega$ .We can easily derive with (2) that

$$a_n x_n = \sum_{k=0}^{n-1} \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_k^{-1} a_n Q_k y_k + \frac{a_n}{q_n} u_k^{-1} Q_n y_n = (Cy)_n \tag{11}$$

where, $C = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} (\frac{1}{q_k} - \frac{1}{q_{k+1}}) u_k^{-1} a_n Q_k, & \text{if } 0 \leq k \leq n-1 \\ \frac{a_n}{q_n} u_k^{-1} Q_n, & \text{if } k = n \\ 0, & \text{if } k > n \end{cases}$$

for all $n, k \in N$. Thus we observe by combining (11) with (i) of Lemma 3.1 that $ax = (a_n x_n) \in l_1$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Cy \in l_1$ whenever $y \in l(p)$. This gives the result that $[r^q(\triangle_u^p)]^\alpha = D_1(u,p)$.

Further, consider the equation,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \left( \frac{a_k}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] y_k = (Dy)_n \tag{12}$$

where, $D = (d_{nk})$ is defined as

$$d_{nk} = \begin{cases} \left( \frac{a_k}{q_k} + (\frac{1}{q_k} - \frac{1}{q_{k+1}}) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}$$

Thus we deduce from Lemma 3.3 with (12) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Dy \in c$ whenever $y \in l(p)$. Therefore,

we derive fom (8) that

$$\sum_k \left| \left[ \left( \frac{a_k}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] B^{-1} \right|^{p_k'} < \infty \tag{13}$$

and $\lim_n d_{nk}$ exists and hence shows that that $[r^q(\triangle_u^p)]^\beta = D_2(u,p) \cap \ cs$.

As this, from Lemma 3.2 together with (12) that $ax = (a_k x_k) \in bs$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Dy \in l_\infty$ whenever $y = (y_k) \in l(p)$. Therefore, we again obtain the condition (13) which means that $[r^q(\triangle_u^p)]^\gamma = D_2(u,p) \cap \ cs$ and the proof of the theorem is complete.

Theorem 3.5 : Let Let $0 < p_k \leq 1$ for every $k \in N$. Define the sets $D_3(u,p)$ and $D_4(u,p)$ as follows

$$D_3(u,p) = \{a = (a_k) \in \omega :$$

$$\sup_{K\in F}\sup_{k}|\sum_{n\in K}[(\frac{1}{q_k}-\frac{1}{q_{k+1}})u_k^{-1}a_nQ_k$$
$$+\frac{a_n}{q_n}u_k^{-1}Q_n]B^{-1}|^{p_k}<\infty\}$$

and

$$D_4(u,p)=\{a=(a_k)\in\omega:$$

$$\sup_k\left|\left[\left(\frac{a_k}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}a_i\right)u_k^{-1}Q_k\right]B^{-1}\right|^{p_k}$$
$$<\infty\}.$$

Then, $[r^q(\triangle_u^p)]^\alpha=D_3(u,p)$ and

$$[r^q(\triangle_u^p)]^\beta=[r^q(\triangle_u^p)]^\gamma=D_4(u,p)\cap\ cs.$$

Proof. This is obtained by proceeding as in the proof of Theorem 2.7, above by using second parts of Lemmas 3.1, 3.2 and 3.3 instead of the first parts. So, we omit the details

Theorem 3.6 : Define the sequence $b^{(k)}(q)=\{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\triangle_u^p)$ for every fixed $k\in N$ by

$$b_n^{(k)}(q)$$
$$=\begin{cases}(\frac{1}{q_n}-\frac{1}{q_{n+1}})u_k^{-1}Q_n+u_k^{-1}\frac{Q_k}{q_k}, & \text{if } 0\le n\le k-1\\ 0, & \text{if } n>k-1.\end{cases}$$

Then, the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r^q(\triangle_u^p)$ and any $x\in r^q(\triangle_u^p)$ has a unique representation of

$$x=\sum_k\lambda_k(q)b^{(k)}(q)\tag{14}$$

where, $\lambda_k(q)=(R^q\triangle x)_k$ for all $k\in N$ and $0<p_k\le H<\infty$.

Proof :It is clear that $b^{(k)}(q)\subset r^q(\triangle_u^p)$, since

$$R^qb^{(k)}(q)=e^{(k)}\in l(p)\ for\ k\in N\tag{15}$$

*and* $0<p_k\le H<\infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in $k^{th}$ place for each $k\in N$.

Let $x\in r^q(\triangle_u^p)$ be given. For every non-negative integer $m$, we put

$$x^{[m]}=\sum_{k=0}^{m}\lambda_k(q)b^{(k)}(q).\tag{16}$$

Then, we obtain by applying $R^q\triangle$ to (16) with (15) that

$$R^q\triangle x^{[m]}=\sum_{k=0}^{m}\lambda_k(q)R^q\triangle b^{(k)}(q)=\sum_{k=0}^{m}(R^qx)_ke^{(k)}$$

and

$$\left(R^q\left(x-x^{[m]}\right)\right)_i=\begin{cases}0, & \text{if } 0\le i\le m\\ (R^q\triangle x)_i, & \text{if } i>m\end{cases}$$

where $i,m\in N$. Given $\varepsilon>0$ , there exists an integer $m_0$ such that

$$\left(\sum_{i=m}^{\infty}|(R^q\triangle x)_i|^{p_k}\right)^{\frac{1}{M}}<\frac{\varepsilon}{2}$$

for all $m\ge m_0$. Hence,

$$h_\triangle\left(x-x^{[m]}\right)=\left(\sum_{i=m}^{\infty}|(R^q\triangle x)_i|^{p_k}\right)^{\frac{1}{M}}$$
$$\le\left(\sum_{i=m_0}^{\infty}|(R^q\triangle x)_i|^{p_k}\right)^{\frac{1}{M}}$$
$$<\frac{\varepsilon}{2}<\varepsilon$$

for all $m\ge m_0$, which proves that $x\in r^q(\triangle_u^p)$ is represented as (14).

Let us show the uniqueness of the representation for $x\in r^q(\triangle_u^p)$ given by (13). Suppose, on the contrary; that there exists a representation $x=\sum_k\mu_k(q)b^k(q)$. Since the linear transformation T from $r^q(\triangle_u^p)$ to $l(p)$ used in the Theorem 2.2 is continuous we have

$$(R^q\triangle x)_n=\sum_k \mu_k(q)\left(R^q\triangle b^k(q)\right)_n$$
$$=\sum_k \mu_k(q)e_n^{(k)}=\mu_n(q)$$

for $n\in N$, which contradicts the fact that $(R^q x)_n=\lambda_n(q)$ for all $n\in N$. Hence, the representation (14) is unique. This completes the proof.

## 4. Matrix Mappings on the Space $r^q(\triangle_u^p)$ :

In this section, we characterize the matrix mappings from the space $r^q(\triangle_u^p)$ to the space $l_\infty$.

Theorem 4.1:(i) Let $1<p_k\le H<\infty$ for every $k\in N$. Then $A\in(r^q(\triangle_u^p):l_\infty)$ if and only if there exists an integer $B>1$ such that

$$C(B)=$$
$$\sup_n\sum_k\left|\left[\frac{a_{nk}}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}a_{ni}\right]u_k^{-1}B^{-1}Q_k\right|^{p_k'} \tag{17}$$

and $\{a_{nk}\}_{k\in N}\in cs$ for each $n\in N$.

(ii) Let $0<p_k\le 1$ for every $k\in N$. Then $A\in(r^q(\triangle_u^p):l_\infty)$ if and only if

$$\sup_{n,k}\left|\left[\frac{a_{nk}}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{n}a_{ni}\right]u_k^{-1}Q_k\right|^{p_k} \tag{18}$$

and $\{a_{nk}\}_{k\in N}\in cs$ for each $n\in N$.

Proof : We only prove the part (i) and (ii) may be proved in a similar fashion. So, let $A\in(r^q(\triangle_u^p):l_\infty)$ and $1<p_k\le H<\infty$ for every $k\in N$. Then $Ax$ exists for $x\in r^q(\triangle_u^p)$ and implies that $\{a_{nk}\}_{k\in N}\in\{r^q(\triangle_u^p)\}^\beta$ for each $n\in N$. Hence necessity of (17) holds.

Conversely, suppose that the necessities (17) hold and $x\in r^q(\triangle_u^p)$,
since $\{a_{nk}\}_{k\in N}\in\{r^q(\triangle_u^p)\}^\beta$ for every fixed $n\in N$, so the $A$-transform of $x$ exists. Consider the following equality obtained by using the relation (11) that
$\sum_{k=0}^m a_{nk}x_k$

$$=\sum_{k=0}^m\left[\frac{a_{nk}}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{m}a_{ni}\right]u_k^{-1}Q_k y_k \tag{19}$$

Taking into account the assumptions we derive from (19) as $m\to\infty$ that

$$\sum_k a_{nk}x_k$$
$$=\sum_k\left[\frac{a_{nk}}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{\infty}a_{ni}\right]u_k^{-1}Q_k y_k \tag{20}$$

Now, by combining (20) and the inequality which holds for any $B>0$ and any complex numbers $a,b$

$$|ab|\le B\left(\left|aB^{-1}\right|^{p'}+|b|^p\right)$$

with $p^{-1}+p'^{-1}=1$( see [10]), one can easily see that

$$\sup_{n\in N}\left|\sum_k a_{nk}x_k\right|$$
$$\le\sup_{n\in N}\sum_k\left|\left[\frac{a_{nk}}{q_k}+\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)\sum_{i=k+1}^{\infty}a_{ni}\right]u_k^{-1}Q_k\right||y_k|$$
$$\le B\left[C(B)+h_1^B(y)\right]<\infty.$$

This shows that $Ax\in l_\infty$ whenever $x\in r^q(\triangle_u^p)$. This completes the proof.

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