Counterparty Credit Risk evaluation for Accumulator derivatives: the Brownian Local Time approach

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Abstract: In this paper we aim at exploiting the properties of the *Brownian Local Time* to estimate the Counterparty Credit Risk for a specific class of financial derivatives, i.e. the so called *Accumulator derivatives*, within a Black and Scholes-type market. The comparison with the results obtained by made use of a standard Monte Carlo approach, clearly shows the superiority of our proposal, which runs in smaller execution times and with better estimation accuracy.

Key–Words: Counterparty Credit Risk, Brownian Local Time, Accumulator Derivatives

1 Introduction

The last decades have seen a rapid and significant increase of *Over-the-Counter* (OTC) contracts, that is, financial bilateral agreements among parties, or their intermediaries, without the supervision of an exchange. Such a boost depends on several causes, especially a considerable flexibility in setting the terms of the contracts, which allows an augmented market liquidity, as well as a strong adaptability to the needs of small companies, that would not be able to verify all the requirements to be included in the exchange listings.

On the other hand, these derivatives contracts have a risk profile in which the counterparty credit merit may play a crucial role, as seen after the financial crisis in 2008. In other words, the OTC contracts expose to the so called *Counterparty Credit Risk,* namely, the risk that a counterparty in a financial transaction will default, before the final settlement of the transaction's cash flows, see [1], [2].

Standard techniques for the evaluation of such an exposure are based on classical Monte Carlo methods, which are characterized by a strong dependence on the number of considered assets and related high computational time costs, see, e.g., [8]. As an example, a medium bank requires $D = O(10^4)$ derivative deals and $U = O(10^3)$ risk factors, evaluated in $K = 20$ time steps with $N = 2000$ simulations, which need $K \cdot N \cdot U = 4 \cdot 10^7$ grid points *at first step*, $K \cdot N \cdot D = 4 \cdot 10^8$ tasks *at second step*, etc.

In the present paper we propose an alternative approach, based on the theory of Brownian Local Times, to estimate the Counterparty Credit Risk (CCR) exposure, also providing a concrete example of its performance when an exotic path-dependent derivatives option is taken into account. In particular we apply our methods to the CCR related to the *Accumulator* type derivatives, whose payoff depends on the time spent by the underlying below or above a given level, between two boundaries or outside of them, see, e.g., [9]. We would also like to underline that similar computational problems also arise when considering very simple financial contracts, as in the case of the European options, or when considering standard interest rate models, e.g. the CoxIngersollRoss (CIR) model, or constant elasticity of variance (CEV) model. Concerning latter cases, a powerful alternative to the Monte Carlo and enhanced Monte Carlo approaches, has been provided by the *Polynomial Chaos Expansion* technique, see, e.g., [3], and references therein.

Therefore, we first consider the theoretical properties of the aforementioned Brownian process, which is defined as follows

$$L_t(a) = \frac{1}{2\varepsilon} \lim_{\varepsilon \to 0} \mu\{0 \leq s \leq t, : |W_s - a| \leq \varepsilon\} , \tag{1}$$

for $t \in [0, T]$, $a \in \mathbb{R}$ and μ is the Lebesgue measure, indicating the amount of time spent by the Brownian Motion $\{W_t\}_{t\geq 0}$, close to a given point $a \in \mathbb{R}$.

2 The mathematical framework

Introduced for the first time in the literature in 1948 by Paul Lévy, see [7], in terms of *mesure de voisinage* as in Eq. (1), the Brownian Local Time, BLT from now on, was thoroughly studied from a mathematical point of view by several authors in the early nineties. In particular, Takacs, see [10], states that the random field

$$\{L_t(x,\omega) \,:\, (t,x) \in [0,T]\times\mathbb{R},\, \omega \in \Omega\}$$

is called a *Brownian Local Time* if the random variable $L_t(x)$ is $\mathfrak{F}$-measurable, the function $(t,x) \longmapsto L_t(x,\omega)$ results to be continuous and

$$\Gamma_t(B,\omega) := \int_0^t \mathbb{1}_B(W_s)\mathrm{d}s = \int_B L_t(x,\omega)\mathrm{d}x\ , \quad (2)$$

for $0 \leq t \leq \infty$, $B \in \mathcal{B}(\mathbb{R})$.

The existence of such a stochastic process is given, e.g., in [5]. In [4], the authors give a representation of BLT in terms of probability density, i.e.

$$\begin{aligned} &P(L(t,a) \in \mathrm{d}y) = f(y;t,a,\sigma,\nu,S) \\ &= \sqrt{\frac{2}{\pi t}}\sigma a \left(\frac{a}{S}\right)^{\nu} e^{-\nu^2\sigma^2\frac{t}{2} - \frac{(\sigma^2 ay + |\log(a/S)|)^2}{2\sigma^2 t}} \\ &+ |\nu|\sigma^2 a\left(\frac{a}{S}\right)^{\nu}\Big[e^{-|\nu|(\sigma^2 ay + |\log(a/S)|)} \\ &\times Erfc\left(\frac{\sigma^2 ay + |\log(a/S)|}{\sigma\sqrt{2t}} - |\nu|\sigma\sqrt{\frac{t}{2}}\right) \\ &- e^{|\nu|(\sigma^2 ay + |\log(a/S)|)} \\ &\times Erfc\left(\frac{\sigma^2 ay + |\log(a/S)|}{\sigma\sqrt{2t}} + |\nu|\sigma\sqrt{\frac{t}{2}}\right)\Big]\ , \end{aligned} \quad (3)$$

where t represents the time up to which the BLT is evaluated, a is the underlying, σ is the volatility parameter, $\nu := -\frac{1}{2} + \frac{r}{\sigma^2}$, r being the risk-free rate, S represents the spot price, and $Erfc(z)$ is the complementary error function, namely

$$\begin{aligned} &Erfc(z) = 1 - Erf(z), \\ &\text{with } Erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\mathrm{d}x\ . \end{aligned}$$

3 The proposal

We refer to a Black and Scholes financial market model, consisting of a riskless security and a risky asset, namely we consider the following set of equations

$$\begin{cases} \mathrm{d}B_t &= rB_t\mathrm{d}t \\ \mathrm{d}S_t &= S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t \end{cases}, \quad (4)$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\{W_t\}_{t\geq 0}$ represents a standard Brownian motion.

In such a market we also consider an *Accumulator derivative,* a structured financial product which is sold by a *seller* to an investor, requiring the issuer to sell shares of some underlying security at a predetermined strike price.

It follows that accumulators do not give the option to either party to refrain from exercising. The strike price is typically settled on a periodical basis and investors are said to be accumulating holdings in the underlying stock over time to maturity. As an example, consider the Financial Times Stock Exchange (FTSE) and the particular accumulator derivative known as *Income Accumulator Plan 6.*, which is defined by the following payoff

$$Payoff = \begin{cases} 0, & \text{if } \max_{0\leq\tau\leq t_k} S_\tau \geq H \\ (S_{t_k} - K), & \text{if } \max_{0\leq\tau\leq t_k} S_\tau < H,\ S_\tau \geq H \\ 2(K - S_{t_k}), & \text{if } \max_{0\leq\tau\leq t_k} S_\tau < H,\ S_\tau < H \end{cases}, \quad (5)$$

where t_k, $k = 1,\ldots,N$, represent N observation times, which are fixed by the contract, S_{t_k} is the underlying at time t_k, K is the strike price, while H describe the possible knock-out barrier level. Thus, the Fair Value at time t_k, $k = 1,\ldots,N$, reads as follows

$$\begin{aligned} FV_{t_k} = \sum_{j=1}^{N} &[C(S_0,K,T-t_j,\sigma,H) \\ &-2P(S_0,K,T-t_j,\sigma,H)]\cdot e^{-r(T-t_k)}\ , \end{aligned} \quad (6)$$

where $C(S_0,K,T-t_j,\sigma,H)$, resp. $P(S_0,K,T-t_j,\sigma,H)$, represents the fair price of a call option, resp. of a put one.

On the other hand, the derivative we are considering, provides, at least, a daily fixing frequency, hence we obtain

$$\begin{aligned} FV_{t_k}^{(LT)} = e^{-r(T-t_k)} &\int_{\mathbb{R}}\int_0^{\infty} yf(y;T,x,\sigma,\nu,S) \\ &\times\left[(x-K)^+ - 2(K-x)^+\right]\mathrm{d}y\mathrm{d}x\ . \end{aligned} \quad (7)$$

The comparison between the benchmark, given by Eq. (6), and our proposal, namely Eq. (7), is shown in the following Table, with respect to different values of the risk-free interest rate r, the strike price K and volatility parameter σ.

r	K	σ	FV BSD	FV LT	$\Delta(LT, BSD)$
0,01	0,9	15%	0,0961	0,0961	0,00%
0,01	0,9	25%	0,0783	0,0781	-0,26%
0,01	1	15%	-0,0323	-0,0322	-0,31%
0,01	1	25%	-0,0587	-0,0576	-1,87%
0,02	0,9	15%	0,1008	0,1008	0,00%
0,02	0,9	25%	0,0837	0,0839	0,24%
0,02	1	15%	-0,0248	-0,0247	-0,40%
0,02	1	25%	-0,0509	-0,0501	-1,57%

The Counterparty Risk estimation procedure

In what follows, we focus our attention on the CCR appraisal, which represents the cornerstone of the our original proposal.

In agreement with the Basel III accord, see [1], the estimate of the CCR is equivalent to assess the extent to which a financial institute may be exposed to a counterparty in case of default. Latter quantity is known as the *Exposure At Default* (EAD).

Among the various metrics described in Basel III for the CCR estimation, we choose the *Expected Exposure* (EE) and the *Expected Positive Exposure* (EPE).

The former is the average of the distribution of exposures at any particular future date before the longest maturity in the portfolio, while the latter is the weighted average over time of the expected exposure, where weights reflect the proportion that an individual expected exposure represents with respect to the entire exposure horizon time interval, see, e.g., [11], namely

$$EE_{t_k} = \frac{1}{N} \sum_{n=1}^{N} MtM\left(t_k, S_{k,n}\right)^+ , \; N \in \mathbb{N}^+ , \quad (8)$$

$$EPE = \frac{1}{T} \sum_{k=1}^{K} EE_k \cdot \Delta_k , \quad (9)$$

where $\Delta_k = t_k - t_{k-1}$ indicates the time interval between two consecutive time buckets at the k-th level, $MtM(t_k, S_{k,n})$ is the fair value of the derivative at time bucket t_k,, with respect to the underlying value S_k.

By exploiting Eq. (3), we have, for all time bucket t_k, $k = 1, \ldots, N$,

$$EE^{(LT)}_{t_k} = \mathbb{E}\left(FV^{(LT)}_{t_k}\right) \quad (10)$$

$$EPE^{(LT)} = \frac{1}{T} \int_0^T \int_{\mathbb{R}} e^{-r(T-t)} \mathbb{E}(L(T, x)) \times [(x-K)^+ - 2(K-x)^+] \mathrm{d}x\mathrm{d}t . \quad (11)$$

In the following Table we show the behaviour of our approach against the classical Monte Carlo method, for different values of the strike price K, the risk-free interest rate r, and the volatility parameter σ.

(K, σ, r)	$EPE^{(BS)}$	$EPE^{(LT)}$	$\Delta(LT, BS)$
$(4.78, 0.15, 0.01)$	0,9303454395	0,9303781163	0,00351%
$(4.78, 0.2, 0.01)$	0,9049015095	0,9048279190	-0,00813%
$(4.78, 0.3, 0.01)$	0,8251642939	0,8247838254	-0,04611%
$(3.75, 0.15, 0.01)$	1,9686102762	1,9686848833	0,00379%
$(3.75, 0.2, 0.01)$	1,9675941521	1,9676336107	0,00201%
$(3.75, 0.3, 0.01)$	1,9547899122	1,9547735333	-0,00084%
$(2.98, 0.15, 0.01)$	2,7348505526	2,7349168463	0,00242%
$(2.98, 0.2, 0.01)$	2,7348375084	2,7348585689	0,00077%
$(2.98, 0.3, 0.01)$	2,7336498018	2,7336545073	0,00017%
$(4.78, 0.15, 0.02)$	0,9556220367	0,9558308683	0,02185%
$(4.78, 0.2, 0.02)$	0,9318141129	0,9318254905	0,00122%
$(4.78, 0.3, 0.02)$	0,8548444479	0,8542861489	-0,06531%
$(3.75, 0.15, 0.02)$	1,9871484085	1,9873803563	0,01167%
$(3.75, 0.2, 0.02)$	1,9862637277	1,9864841609	0,01110%
$(3.75, 0.3, 0.02)$	1,9743416072	1,9744396571	0,00497%
$(2.98, 0.15, 0.02)$	2,7496039372	2,7497784936	0,00635%
$(2.98, 0.2, 0.02)$	2,7495932160	2,7497625846	0,00616%
$(2.98, 0.3, 0.02)$	2,7485174039	2,748661624	0,00525%

4 Conclusion

The methodology we propose in the present work is founded on the possibility of expressing the Brownian Local Time in terms of its probability density. The implementation of this approach, with regard to EPE evaluation, leads to numerical results that show a twofold advantage over the use of standard Monte Carlo techniques, namely: first it reduces the execution time, then it improves the *EE* appraisal accuracy.

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