Exit times and places for a Wiener process with sign-dependent exponential jumps

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Abstract: We consider a one-dimensional jump-diffusion process $\{X(t), t \geq 0\}$ whose continuous part is a Wiener process with zero drift. The jumps are exponential and depend on the sign of $X(t)$. Let $\tau(x)$ be the first time that the process, starting from $X(0) = x$, is equal to zero, or $|X(t)| = d$. We obtain exact analytical expressions for the moment-generating function of $\tau(x)$, its mean, and the probability that $X(\tau(x)) = 0$. To do so, we solve integro-differential equations, subject to the appropriate boundary conditions.

Key-Words: Hitting time, Brownian motion, Poisson process, jump size, integro-differential equation.

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1 Introduction

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. The two stochastic processes are assumed to be independent. We define the one-dimensional jump-diffusion process $\{X(t), t \geq 0\}$ by

$$X(t) = X(0) + \mu \, \mathrm{d}t + \sigma \, \mathrm{d}B(t) + \sum_{i=1}^{N(t)} Y_i, \qquad (1)$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants. If the rate λ is equal to 0, $\{X(t), t \geq 0\}$ is a Wiener process with drift μ and dispersion parameter σ. Moreover, $Y_1, Y_2, \ldots$ are independent random variables that are identically distributed as the variable Y whose probability density function is given by

$$f_Y(y) = \begin{cases} \theta e^{-\theta y} & \text{for } y \geq 0 \text{ if } X(t) < 0, \\ \theta e^{\theta y} & \text{for } y < 0 \text{ if } X(t) > 0, \end{cases} \qquad (2)$$

where $\theta > 0$. That is, when $X(t)$ is negative, the jumps are positive and exponentially distributed with parameter θ, whereas when $X(t)$ is positive, the jumps are negative and their absolute values are exponentially distributed with parameter θ.

In, [1] and, [2] the random variable Y had an asymmetric double exponential distribution, irrespective of the value of $X(t)$. Furthermore in, [2] the jump-diffusion process $\{X(t), t \geq 0\}$ was generalized as follows:

$$\begin{aligned} X(t) &= X(0) + \int_0^t \mu[X(s)] \, \mathrm{d}s \\ &\quad + \int_0^t \sigma[X(s)] \, \mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i, \end{aligned} \qquad (3)$$

where $\mu(\cdot) \in \mathbb{R}$ and $\sigma(\cdot) > 0$.

Next, assume that $X(0) = x$. In, [1] the authors defined the first-passage time

$$T_b(x) = \inf\{t \geq 0 : X(t) \geq b \; (> 0)\}, \qquad (4)$$

where $x \leq b$, and obtained various results about the distribution of $T_b(x)$ and $X(T_b(x))$. This random variable was generalized in, [2] to

$$T_{a,b}(x) := \inf\{t \geq 0 : X(t) \notin (a, b)\} \qquad (5)$$

for $x \in [a, b]$.

In this paper, we take $\mu = 0$ in Eq. (1). Moreover, we define

$$\tau(x) = \inf\{t \geq 0 : X(t) = 0 \text{ or } |X(t)| \geq d\}, \qquad (6)$$

where $x \in [-d, d]$.

In the next section, we will obtain exact analytical expressions for the moment-generating function of $\tau(x)$:

$$M(x; \alpha) := E\left[e^{-\alpha \tau(x)}\right], \qquad (7)$$

where $\alpha > 0$, its mean $m(x) := E[\tau(x)]$, and the probability $p(x) := P[X(\tau(x)) = 0]$. As we will see, the functions $M(x;\alpha)$, $m(x)$ and $p(x)$ satisfy integro-differential equations. We will solve these equations, subject to the appropriate boundary conditions, by first transforming them into ordinary differential equations.

In general, boundary value problems for integro-differential equations are very difficult to solve explicitly and exactly. In addition to, [2] the author has treated such problems in a number of papers, as well as stochastic control problems involving jump-diffusion processes. In, [3] the first-passage area of one-dimensional jump-diffusion processes was computed; see also, [4], [5], [6] and, [7].

Jump-diffusion processes appear in various fields. An important application is in mathematical finance ([8]). In, [9] they are used as models in reliability theory.

2 Explicit solutions

The infinitesimal generator of the jump-diffusion process defined in Eq. (1) with $\mu = 0$ is (see, [1] or, [10])

$$\mathcal{L}u(x) := \frac{1}{2}\sigma^2 u''(x) + \lambda\left\{\int_{-\infty}^{\infty} u(x+y)\, f_Y(y)\,\mathrm{d}y - u(x)\right\} \tag{8}$$

for any twice continuously differentiable function $u(x)$. It follows that the function $M(x;\alpha)$ satisfies the integro-differential equation (IDE) (dropping the argument α for the sake of brevity)

$$\frac{1}{2}\sigma^2 M''(x) + \lambda\left\{\int_{-\infty}^{\infty} M(x+y)\, f_Y(y)\,\mathrm{d}y - M(x)\right\} = \alpha M(x) \tag{9}$$

and is such that $M(0) = 1$ and $M(x) = 1$ if $|x| \geq d$. Moreover, by symmetry, we can write that $M(-x) = M(x)$. It follows that we can consider the above equation in the interval $[0, d]$ only.

Next, with the function $f_Y(y)$ defined in Eq. (2), if $x > 0$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} M(x+y)\, f_Y(y)\,\mathrm{d}y &= \int_{-\infty}^{0} M(x+y)\,\theta\, e^{\theta y}\,\mathrm{d}y \\ &= \int_{-\infty}^{-x-d} 1\cdot\theta\, e^{\theta y}\,\mathrm{d}y + \int_{-x-d}^{0} M(x+y)\,\theta\, e^{\theta y}\,\mathrm{d}y \\ &= e^{-\theta(x+d)} + \int_{-x-d}^{0} M(x+y)\,\theta\, e^{\theta y}\,\mathrm{d}y \\ &\overset{z=x+y}{=} e^{-\theta(x+d)} + \int_{-d}^{x} M(z)\,\theta\, e^{\theta(z-x)}\,\mathrm{d}z. \end{aligned} \tag{10}$$

Differentiating Eq. (9) and making use of Leibniz integral rule together with the above result, we obtain that

$$\frac{1}{2}\sigma^2 M'''(x) - (\lambda+\alpha)\, M'(x) + \lambda\theta\left\{M(x) - e^{-\theta(x+d)} - \int_{-d}^{x} M(z)\,\theta\, e^{\theta(z-x)}\,\mathrm{d}z\right\} = 0. \tag{11}$$

Finally, since (from Eqs. (9) and (10))

$$\lambda\left\{e^{-\theta(x+d)} + \int_{-d}^{x} M(z)\,\theta\, e^{\theta(z-x)}\,\mathrm{d}z\right\} = -\frac{1}{2}\sigma^2 M''(x) + (\lambda+\alpha)\, M(x), \tag{12}$$

we can state the following proposition.

Proposition 2.1. *The moment-generating function $M(x)$ of the random variable $\tau(x)$ satisfies, for $\lambda > 0$, the third-order linear ordinary differential equation (ODE)*

$$\frac{1}{2}\sigma^2 M'''(x) + \frac{1}{2}\theta\sigma^2 M''(x) - (\lambda+\alpha)\, M'(x) = \alpha\theta\, M(x) \tag{13}$$

for $x \in (0, d)$. Furthermore, the boundary conditions are $M(0) = 1$ and $M(x) = 1$ if $x \geq d$.

Remark 2.1. *If the stochastic process jumps from a positive to a negative value (or vice versa), we assume that $X(t)$ was never equal to 0 when the jump occurred.*

We will now solve Eq. (13). For the sake of simplicity, we set $\sigma = \lambda = \theta = \alpha = d = 1$. The equation reduces to

$$\frac{1}{2}M'''(x) + \frac{1}{2}M''(x) - 2M'(x) = M(x). \tag{14}$$

With the help of the software program *Maple*, we find that the solution to the above equation that satisfies the conditions $M(0) = M(1) = 1$ and $M(1/2) = r$ is given by

$$\begin{aligned} M(x) \approx \kappa\Big[&(-3.8292 + 6.0365r)\, e^{-0.4707x} \\ &+ (4.9859 - 5.5079r)\, e^{-2.3429x} \\ &+ (0.5980 - 0.5285r)\, e^{1.8136x}\Big] \end{aligned} \tag{15}$$

for $x \in [0, 1]$, where

$$\kappa := 0.5699. \tag{16}$$

Substituting the above expression for $M(x)$ into the IDE (9), we find that this equation is satisfied if and only if we take $r \approx 0.8239$.

Let

$$\begin{aligned} H_1(x) &:= \frac{1}{2}M''(x) + \lambda\left\{\int_{-\infty}^{\infty} M(x+y)\, f_Y(y)\,\mathrm{d}y \right. \\ &\quad \left. - M(x)\right\} - M(x) \\ &\overset{\lambda=1}{=} \frac{1}{2}M''(x) + \int_{-\infty}^{\infty} M(x+y)\, f_Y(y)\,\mathrm{d}y \\ &\quad - 2M(x) \end{aligned} \tag{17}$$

for $x \geq 0$. The function $H_1(x)$ is shown in Figure 1. We see that it is practically equal to zero in the interval $[0,1]$.

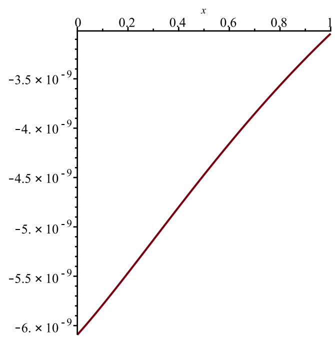

Figure 1: Function $H_1(x)$ defined in Eq. (17) for x in the interval $[0,1]$.

When there are no jumps, that is, when $\lambda = 0$, the function $M_0(x)$ that corresponds to $M(x)$ satisfies the simple second-order linear ODE

$$\frac{1}{2}\sigma^2 M_0''(x) = \alpha M_0(x), \tag{18}$$

subject to the boundary conditions $M(0) = M(d) = 1$. We find, with $\sigma = \alpha = 1$, that

$$M_0(x) = \frac{(e^{\sqrt{2}} - 1)\,e^{-\sqrt{2}x} - (e^{-\sqrt{2}} - 1)\,e^{\sqrt{2}x}}{e^{\sqrt{2}} - e^{-\sqrt{2}}} \tag{19}$$

for $0 \leq x \leq 1$. The functions $M(x)$ and $M_0(x)$ are displayed in Figure 2.

Next, the mean $m(x)$ of the first-passage time $\tau(x)$ is a solution of

$$\begin{aligned} &\frac{1}{2}\sigma^2 m''(x) + \lambda\left\{\int_{-\infty}^{\infty} m(x+y)\, f_Y(y)\,\mathrm{d}y \right. \\ &\left. - m(x)\right\} = -1. \end{aligned} \tag{20}$$

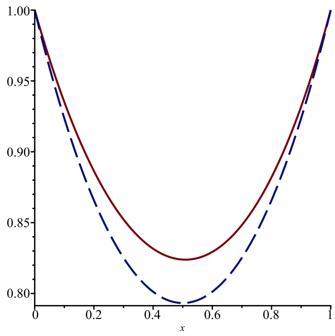

Figure 2: Functions $M(x)$ (full line) and $M_0(x)$ in the interval $[0,1]$, when $\sigma = \lambda = \theta = \alpha = d = 1$.

In addition, the boundary conditions are $m(0) = 0$ and $m(x) = 0$ if $|x| \geq d$. Proceeding as above, we can prove the following corollary.

Corollary 2.1. *The mean $m(x)$ of the random variable $\tau(x)$ satisfies, for $\lambda > 0$, the ODE*

$$\frac{1}{2}\sigma^2 m'''(x) + \frac{1}{2}\theta\sigma^2 m''(x) - \lambda m'(x) = -1 \tag{21}$$

for $x \in (0,d)$, subject to the boundary conditions $m(0) = 0$ and $m(x) = 0$ if $x \geq d$.

Remark 2.2. *Notice that Eq. (21) is a second-order linear ODE with constant coefficients for $l(x) := m'(x)$.*

The solution to Eq. (21) such that $m(0) = m(d) = 0$ and $m(d/2) = r$, when $\sigma = \lambda = \theta = d = 1$, is found to be

$$\begin{aligned} m(x) &\approx x - 0.9520\,(3.4366r - 0.4208)\,e^{-2x} \\ &\quad - 0.0474\,e^{x}\,(34.7345r + 8.0257) \\ &\quad + 4.9177r - 0.02024. \end{aligned} \tag{22}$$

Moreover, the above function satisfies the IDE (20) with the previous choices for the various parameters if the constant r is taken to be $r \approx 0.1754$. Indeed, if we define

$$\begin{aligned} H_2(x) &= \frac{1}{2}m''(x) + \int_{-\infty}^{\infty} m(x+y)\, f_Y(y)\,\mathrm{d}y \\ &\quad - m(x) + 1, \end{aligned} \tag{23}$$

we find that $H_2(x) \approx 0$ for $x \in [0,1]$; see Figure 3.

In the absence of jumps, the function $m_0(x)$ corresponding to $m(x)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 m_0''(x) = -1 \tag{24}$$

and is subject to the conditions $m_0(0) = m_0(1) = 0$. When $\sigma = 1$, we easily find that

$$m_0(x) = x - x^2 \quad \text{for } x \in [0,1]. \tag{25}$$

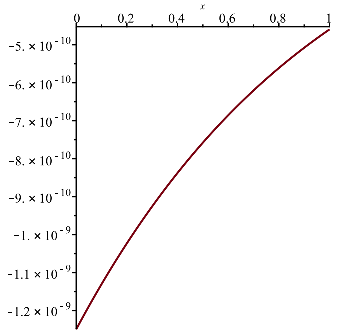

Figure 3: Function $H_2(x)$ defined in Eq. (23) for x in the interval $[0, 1]$.

In Figure 4, we see the effect of the jumps on the mean $m(x)$ of $\tau(x)$.

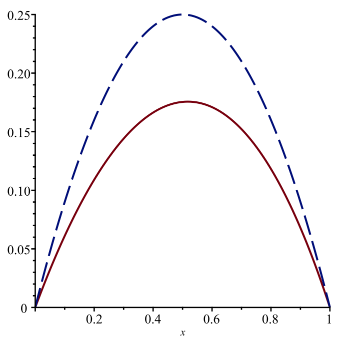

Figure 4: Functions $m(x)$ (full line) and $m_0(x)$ in the interval $[0, 1]$, when $\sigma = \lambda = \theta = d = 1$.

Finally, to obtain the probability

$$p(x) := P[X(\tau(x)) = 0], \tag{26}$$

we must solve the IDE

$$\frac{1}{2}\sigma^2 p''(x) + \lambda \left\{ \int_{-\infty}^{\infty} p(x+y)\, f_Y(y)\, \mathrm{d}y - p(x) \right\} = 0. \tag{27}$$

This function is such that $p(0) = 1$ and $p(x) = 0$ if $|x| \geq d$.

As we did previously, we can transform the above IDE into an ODE.

Corollary 2.2. *The probability $p(x)$ defined in Eq. (26) satisfies, when $\lambda > 0$, the ODE*

$$\frac{1}{2}\sigma^2 p'''(x) + \frac{1}{2}\theta\sigma^2 p''(x) - \lambda p'(x) = 0 \tag{28}$$

for $x \in (0, d)$. The boundary conditions are $p(0) = 1$ and $p(x) = 0$ if $x \geq d$.

When $\sigma = \lambda = \theta = 1$, the ODE in Eq. (28) becomes

$$\frac{1}{2}p'''(x) + \frac{1}{2}p''(x) - p'(x) = 0. \tag{29}$$

As in the case of the function $m(x)$, the above equation is a second-order (homogeneous) ODE for $q(x) := p'(x)$. Its general solution is

$$p(x) = c_1 + c_2 e^{x} + c_3 e^{-2x}, \tag{30}$$

where c_i is an arbitrary constant, for $i = 1, 2, 3$. The solution for which $p(0) = 1$, $p(1) = 0$ and $p(1/2) = r$ is the following:

$$\begin{aligned} p(x) \approx\ & 0.1672\, e^{-2x} \big[(r-1)\, e^{2x+0.5} \\ & + (r-1)\, e^{2x+1.5} + (-r+1)\, e^{3x+0.5} \\ & + e^{2x+2.5} r - e^{3x+1.5} r \\ & + (r-1)\, e^{2x+1} + e^{2x+2} r \\ & - e^{3x+1} r + (-r+1)\, e^{3x} \\ & + e^{2x} r - 19.5716 r + 12.1825 \big] \end{aligned} \tag{31}$$

for $x \in [0, 1]$.

The only constant r such that the function defined in Eq. (31) is a solution of the IDE (27) (when $\sigma = \lambda = \theta = d = 1$) is $r \approx 0.5712$. In Figure 5, we present the function

$$H_3(x) := \frac{1}{2}p''(x) + \int_{-\infty}^{\infty} p(x+y)\, f_Y(y)\, \mathrm{d}y - p(x). \tag{32}$$

It is practically equal to zero, as it should be.

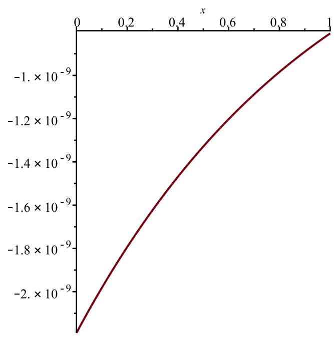

Figure 5: Function $H_3(x)$ defined in Eq. (32) for x in the interval $[0, 1]$.

When $\lambda = 0$ and $\sigma = 1$, the function $p_0(x)$ that corresponds to $p(x)$ satisfies the elementary ODE

$$\frac{1}{2}p_0''(x) = 0, \tag{33}$$

subject to the conditions $p_0(0) = 1$ and $p_0(1) = 0$. The solution is

$$p_0(x) = 1 - x \quad \text{for } x \in [0, 1]. \tag{34}$$

The functions $p(x)$ and $p_0(x)$ are markedly different, as can be seen in Figure 6.

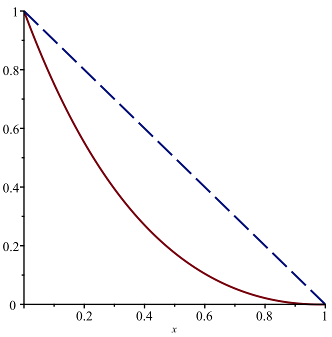

Figure 6: Functions $p(x)$ (full line) and $p_0(x)$ in the interval $[0, 1]$, when $\sigma = \lambda = \theta = d = 1$.

3 Conclusion

Because diffusion processes are unable to adequately reproduce phenomena such as variations in stock market indices like the NASDAQ, it is now common practice to add jumps according to a Poisson process to the model.

When the jump size is a continuous random variable, this has the consequence of transforming ordinary differential equations into integro-differential equations, which makes the problem of calculating quantities of interest such as the average time required for the process to leave a given interval much more difficult. In the case of random jumps having a distribution of discrete type, the equations to be solved would be difference-differential equations.

In this paper, we considered a Wiener process with jumps whose amplitude, in absolute value, was distributed according to an exponential law. However, the sign of the jumps depended on the sign of the stochastic process.

We have obtained analytical expressions for the moment-generating function and the mean of a certain first-passage time. We have also calculated the probability of the process hitting the origin before a boundary at $x = d$.

As a follow-up to this work, we could try to calculate quantities of interest for other processes whose continuous parts are important diffusion processes such as geometric Brownian motion, which is extensively used in financial mathematics.

Finally, we could also consider jump-diffusion processes in two or more dimensions. By using symmetry in the problems studied, it is sometimes possible to reduce these multi-dimensional problems to one-dimensional ones.

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