A Markov Chain Model for the APOS/ACE Instructional Treatment of Mathematics

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Abstract: - The ACE teaching style is the pedagogical approach of the APOS instructional treatment of mathematics implemented with the help of computers. In this paper a Markov Chain model is introduced on the components of the ACE cycle on the purpose of studying mathematically its flow-diagram. This leads to a measure evaluating the student difficulties in learning mathematics. A classroom example is also presented illustrating the applicability and usefulness of the model.

Key-Words: - APOS theory, ACE teaching style of mathematics, Markov Chain (MC), Absorbing MC, Fundamental matrix, Derivative.

1 Introduction

The *APOS/ACE* instructional treatment of mathematics has been developed in the USA during the 1990's by a team of mathematicians and mathematics educators led by Ed Dubinsky [1-3]. In earlier works we have applied the APOS/ACE approach for teaching the irrational numbers [4] and the polar coordinates on the plane [5-6] and also for assessing, with the help of fuzzy logic, its effectiveness in improving the student learning skills [7].

In the present work we introduce a *Markov Chain (MC)* with states the components of the ACE teaching style of mathematics on the purpose of studying mathematically the flow-diagram of the ACE cycle. This leads to a measure for evaluating the student difficulties in learning mathematics. The rest of the paper is formulated as follows: In Section 2 a brief account of the main ideas of the APOS/ACE theory is presented. Our MC model is developed in Section 3, while in Section 4 examples are provided on teaching the derivative illustrating the model's applicability and usefulness in practice. The article closes with the conclusions and some hints for future research on the subject, which are contained in Section 5.

2 The APOS/ACE Instructional Treatment of Mathematics

APOS is a theory based on Piaget's principle that an individual learns by applying certain mental mechanisms to build specific mental structures and utilizes those structures to deal with problems connected to the corresponding situations [8] As a matter of fact, the APOS theory argues that the teaching and learning of mathematics should be based on helping students to use the mental structures that they already have and to develop new, more powerful structures, for handling more and more advanced mathematics. Those structures include *Actions*, *Processes*, *Objects* and *Schemas*, the acronym APOS being formed by the initial letters of the above four words.

Two are the mental mechanisms involved in the APOS approach, called *interiorization* and *encapsulation* respectively. A mathematical concept is first formed as an action. As one repeats and reflects on an action, this action may be interiorized to a process enabling the individual to perform the same activities in his/her mind. When the individual becomes aware of a process as a totality and becomes able to construct transformations on this totality, then the process has been encapsulated to an object. This is often neither easy nor immediate, because encapsulation entails a radical sift in the nature of one's conceptualization, since it signifies the ability to think of the same concept as a mathematical entity to which new, higher-level transformations can be applied. On the other hand, the mental process that led to a mental object through encapsulation remains still available and many mathematical situations require one to *de-encapsulate* an object back to the process that led to it. Finally, the actions, processes and objects involved in a mathematical topic need to be organized in an individual's coherent cognitive schema.

For example, if one can think of a function only through an explicit expression connecting the two variables involved, then he/she is having an action understanding of functions. On the contrary, a process understanding of a function enables the individual to think about it in terms of inputs and outputs, possibly unspecified. Further, an object understanding allows one to form sets of functions, to define operations on such sets, to equip them with a topology, etc. Going back from a composite function to its component functions for the better understanding of the rule of derivation of a composite function, or going back from the derivative to the initial function in order to understand the process of the integration of a function, constitute classical examples of de-encapsulating an object back to the process that led to it Finally, it is the schema structure that enables one to see and use a function in a given mathematical or real world situation. Fig. 1, taken from Dubinsky's personal web page [9], represents graphically the APOS approach.

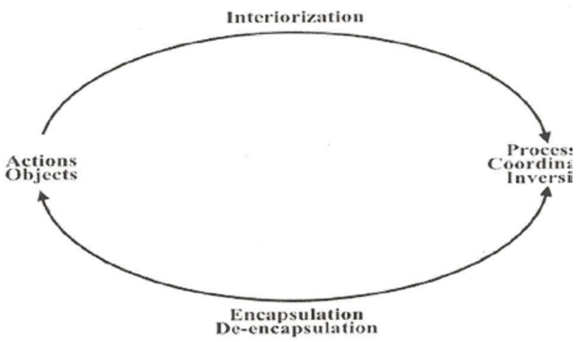


Fig. 1. Graphical representation of the APOS approach

The implementation of the APOS as a framework for teaching and learning mathematics involves three stages. First a theoretical analysis, called *Genetic Decomposition* (GD) of the concepts under study, is performed. In the next stage instructional sequences based on the GD are developed and implemented and finally data are collected and analysed in order to test and refine the GD [2].

The APOS theory has important consequences for education. Simply put, it says that the teaching of mathematics should aim in helping students to use the mental structures they already have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them to build new, more powerful structures for handling more and more advanced mathematics. Dubinsky and his collaborators realized that for each mental construction that comes out of an APOS analysis, one can find a computer task of writing a program or code, such that, if a student engages in that task, he (she) is fairly likely to build the mental construction that leads to learning the corresponding mathematical topic. Based on the above aspect, the pedagogical approach based on APOS analysis, known as the *ACE teaching cycle*, is a repeated cycle of three components: *(A) activities on the computer, (C) Classroom discussion and (E) Exercises done outside the class* (Fig. 2)

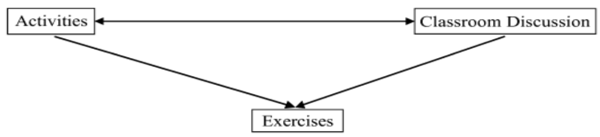


Fig. 2. The ACE teaching cycle.

In applying the ACE cycle the mathematical topic to be learnt is divided to smaller subtopics and each one of the iterations of the cycle corresponds to one of those subtopics. The computer activities, which form the first step of the ACE approach, are designed to foster the students' development of the appropriate mental structures. The students do all of their work in cooperative groups. In classroom the teacher guides the students to reflect on the computer activities and their relation to the mathematical concepts being studied. They do this by performing mathematical skills without using the computer. They discuss their results and listen to explanations by fellow students or the teacher of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems related to the topic being studied. Students reinforce the knowledge obtained in the computer activities and classroom discussions by applying it in solving these problems. The implementation of the ACE cycle and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies of the Dubinsky's team (e.g. [1, 3, 10, 11], etc.).

3 The Markov Chain Model

Roughly speaking, a MC is a stochastic process that moves in a sequence of steps (phases) through a set of states and has a "*one-step memory*". This means that the probability of entering a certain state in a certain step, known as the *transition probability* between steps, depends on the state occupied in the previous step and not in older steps. This is known as the *Markov property*. However, for being able to model as many real life situations as possible by using MCs, one could accept in practice that the transition probability, although it may not be completely independent of previous steps, it mainly depends on the state occupied in the previous step [12]. When the set of states of a MC is a finite set, then we speak about a *finite MC.* For general facts on finite MCs we refer to the book [13]

Here, in order to study mathematically the flow-diagram of the ACE cycle, we introduce a finite MC with states the components S_1 = computer activities, S_2 = classroom discussion and S_3 = homework exercises, of the ACE cycle. Denote by p_{ij} the *transition probability* from state S_i to state S_j, $i, j = 1, 2, 3$. Then the matrix $A = [p_{ij}]$ is called the *transition matrix* of the MC. Taking into account the flow-diagram of the ACE cycle presented in Fig. 2 it is straightforward to check that

$$A = \begin{array}{c c} & \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (1)$$

Since the transition from a state to some other state is the certain event, we have that

$$P_{12} + p_{13} = p_{21} + p_{23} = 1 \qquad (2).$$

A state of a MC is called *absorbing* if, once entered, it cannot be left. Further a MC is said to be an *absorbing MC (AMC)* if it has at least one absorbing state and if from every state it is possible to reach an absorbing state, not necessarily in one step. Obviously the present MC is an AMC with S_3 being its unique absorbing state ad S_1 its starting state.

Applying the standard theory of the AMCs ([13], Chapter 3) we bring the transition matrix A to its *canonical form* A^* by listing the absorbing state first and then we make a partition of A^* as follows:

$$A^* = \begin{array}{c c} & \begin{array}{cccc} S_3 & & S_1 & S_2 \end{array} \\ \begin{array}{c} S_3 \\ \\ S_1 \\ S_2 \end{array} & \begin{bmatrix} 1 & | & 0 & 0 \\ - & - & - & - \\ p_{13} & | & 0 & p_{12} \\ p_{23} & | & p_{21} & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I_1 & | & O \\ - & | & - \\ R & | & Q \end{bmatrix} \qquad (3).$$

In the above partition I_1 is the 1 X 1 unitary matrix, O is a 1 X 2 zero matrix, R is the 2 X 1 transition matrix from the non-absorbing states to the absorbing state and Q is the 2 X 2 transition matrix between the two non absorbing states. Then, if I_2 denotes the 2 X 2 unitary matrix, we have

$$I_2 - Q = \begin{bmatrix} 1 & -p_{12} \\ -p_{21} & 1 \end{bmatrix} \qquad (4).$$

Since the determinant of $I_2 - Q$ is non zero, $I_2 - Q$ is an invertible matrix. Note that that this matrix turns to be always invertible regardless to the number of states of the corresponding AMC [14].

Then, the *fundamental matrix* N of the AMC is defined to be the inverse matrix of $I_2 - Q$. Therefore

$$N = [n_{ij}] = (I_2 - Q)^{-1} = \frac{1}{D\ (I_2 - Q)} adj\ (I_2 - Q) \quad (5).$$

The matrix $adj\ (I_2 - Q)$ in equation (5) is the *adjoin matrix* of $I_2 - Q$ and $D\ (I_2 - Q)$ is the determinant of I_2 - Q.([15], Section 2.4). It is recalled that the adjoin matrix of $I_2 - Q$ is the matrix of the algebraic complements of the *transpose* matrix of $I_2 - Q$, which is obtained by turning the rows of $I_2 - Q$ to columns and vice versa.. Replacing the matrix $I_2 - Q$ from (4) to (5) and making the corresponding calculations one finds that

$$N = \frac{1}{1 - p_{12}p_{21}} \begin{bmatrix} 1 & p_{12} \\ -p_{21} & -1 \end{bmatrix} \qquad (6).$$

It is well known ([13], Chapter 3) that the element n_{ij} of the fundamental matrix N gives the mean number of times in state S_i before the absorption, when the starting state of the AMC is S_j, where S_i and S_j are non absorbing states. In our case, since S_1 is the starting state of the MC, it becomes evident that the mean number of steps of the MC before the absorption is given by the sum

$$t = n_{11} + n_{12} = \frac{1 + p_{12}}{1 - p_{12}p_{21}} \quad (7).$$

It is logical to accept that the greater is the value of t, the more the student difficulties during the ACE cycle. On the other hand, the total time spent is another factor, apart for t, indicating the student difficulties. However, the total duration of the steps S_1 and S_2 of the ACE cycle is usually prefixed by the instructor, which means that in this case t could be considered as a measure of the student difficulties during the computer activities and the classroom discussion.

4 The Classroom application

The following classroom application took place some time ago at the Graduate Technological Educational Institute of Western Greece in the city of Patras with subjects 30 students of the first term in an engineering department of the School of Technological Applications. In order to help students to have a better understanding of the graphical representation of the derivative, we designed [16], in collaboration with Vahid Borji who performed a similar classroom application in an Iranian University [17], an APOS GD by giving emphasis to the following points:

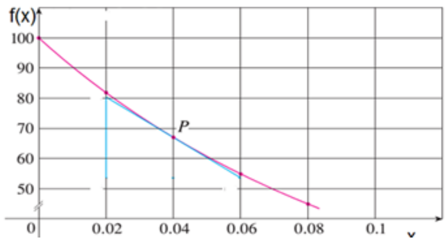
1. Connecting two points (*a*, *f(a)*) and (*b*, *f(b)*) on the graph of a given function *y* = *f(x)* to construct the corresponding chord of the graph.
2. Calculating the slope $\frac{f(b)-f(a)}{b-a}$ of a secant line at the point (*a*, *f(a)*) as the other point (*b*, *f(b)*) is moving approaching it.
3. Defining the tangent line at the point (*a*, *f(a)*) of the graph of a function *y* = *f*(*x*) and calculating its slope by the limit: $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, which is by definition the derivative *f'(a)*.
4. Calculating on the basis of the above process the derivative *f΄'(a)* at a point *(α, f(α))* from a given table of suitable values of the function *y* = *f(x)* without using limits.
5. Presenting examples of constructing the graph of the derivative function *f΄(x)* when the graph but not the analytic formula of *y* = *f(x)* is given.

Next, an ACE approach was developed on the basis of the above GD. Three computer activities were designed with the help of the proper software corresponding to three iterations of the ACE cycle as follows:

- The first activity, connected to the points 1-3 of the above GD, focused on a limit process, where a point B (*b*, *f(b)*) moving on the graph of *y* = *f(x)* approaches the fixed point A (*a*, *f(a)*), which means that the corresponding secant line approaches the tangent line of the graph at the point A.
- In the second activity, connected to the point 4 of the GD, a ready computer procedure was given to students that designs the graph of *y* = *f(x)* and its tangent line at *a*, computes the slope of the tangent and plots the point (*a*, *f΄(α)*) in the same coordinate system.
- The third activity, connected to the point 5 of the GD, expanded the second one to a procedure that plots any points of the form (*x*, *f΄(x)*) when the graph of *y* = *f(x)* is given and designs the graph of the derivative function *f΄(x)* in the same coordinate system

The following three exercises were given to students for solution without the help of computers after the end of each computer activity:

Exercise 1 (connected to the first activity): Using the graph of the function *y* = *f(x)* and the Table of its values given in Fig. 3 approximate the value of the derivative *f ΄(x)* at x = 0.04.



x	f(x)
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Fig. 3**.** The graph and data of Exercise 1

Exercise 2 (Connected to the second computer activity)**:** The line L is the tangent to the graph of the function *y*= *f(x)* of Fig. 4 at the point (4, 4). Calculate the value of f ΄(4).

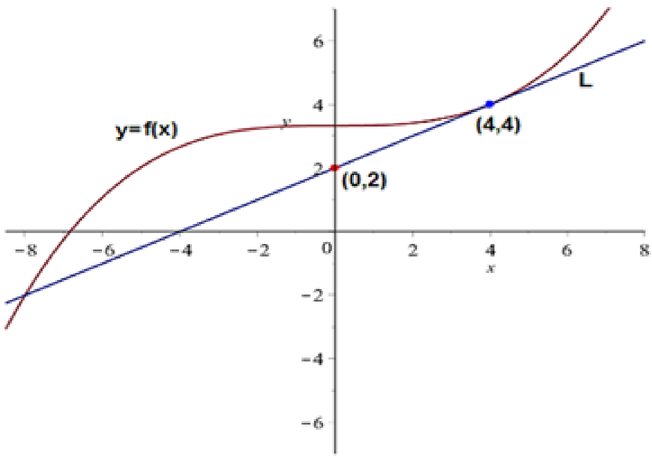


Fig. 4. The graph of *y*= *f(x) in* Exercise 2

Exercise 3 (Connected to the third computer activity)**:** Taking into account that the tangent at the point (a, $f(a)$) of the graph of the function $y = f(x)$ of Fig. 5 is parallel and that the tangent at (b, $f(b)$) is perpendicular to the x-axis, sketch the graph of the derivative function $f'(x)$.

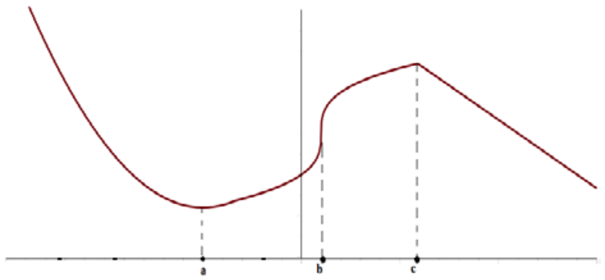


Fig. 5. The graph of $y= f(x)$ in Exercise 3

The last exercise, having the greatest difficulty among the others, can be solved by taking into account the following points, for which a particular emphasis had been given by the instructor during the classroom discussions:

- Since the tangent of the given graph at (a, $f(a)$) is parallel to the x-axis, its slope is equal to zero, which means that $f'(\alpha) = 0$. Consequently, the graph of $f'(x)$ intersects the x-axis at a.
- Since the tangent of the graph of $f(x)$ at b is perpendicular to the x-axis, its slope is equal to $+\infty$, therefore b does not belong to the domain of $f'(x)$.
- At the point (c, $f(c)$) the left and right tangents to the graph of $f(x)$ are different, which means that $f'(x)$ is not defined at c.
- If $f(x)$ is strictly increasing (decreasing) in an interval I, then $f'(x)>0$ (<0) for all x in I. Therefore, the graph of $f'(x)$ in I lies over (under) the x-axis
- If the concavity of $f(x)$ in an interval I is upwards (downwards), which means that $f''(x)>0$ (<0), the derivative function $f'(x)$ is strictly increasing (decreasing) in I.

All the above lead to the draft design of the graph of $f'(x)$ presented in Fig. 6

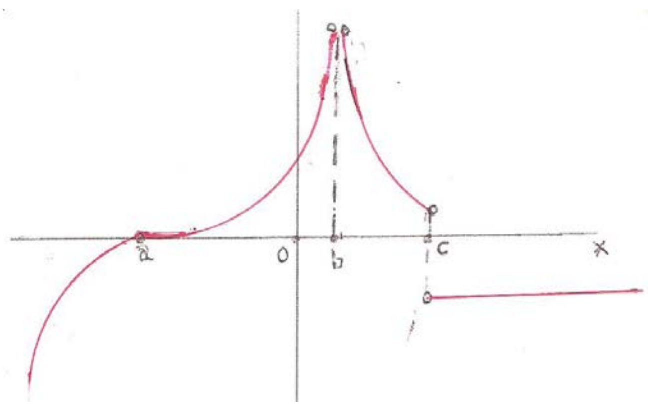


Fig. 6. The graph of $f'(x)$ in Exercise 3

Inspecting the student answers in Exercise 1, I realized that 18 out of 30 solved it correctly. This means that the target of the first iteration of the ACE cycle was succeeded by those students. On the contrary, it became evident that for the rest of the students the classroom discussion following the first computer activity was necessary in order to reflect better on this activity and its relation to the mathematical topic being studied. In other words and in terms of the MC model of Section 2 one concludes that $p_{13} = \frac{18}{30}$ and $p_{12} = \frac{12}{30}$

At the end of the classroom discussion an analogous exercise was given for solution to the 12 students that had failed to solve Exercise 1. In this case I found 8 correct solutions, which means that $p_{23} = \frac{8}{12}$ and $p_{21} = \frac{4}{12}$. Replacing the values of the tradition probabilities in equation (7) one finds that t $= \frac{21}{13} \approx 1.62$.

Working similarly for Exercise 2 and the second iteration of the ACE cycle I found that $p_{13} = \frac{14}{30}$, $p_{12} = \frac{16}{30}$ and $p_{23} = p_{21} = \frac{8}{16}$. In this case equation (7) gives that t = $\frac{23}{11} \approx 2.09$.

Finally, for Exercise 3 and the third iteration of the ACE cycle I found that $p_{13} = \frac{10}{30}$, $p_{12} = \frac{20}{30}$ and $p_{23} = p_{21} = \frac{10}{20}$. In this case equation (7) gives that t $= \frac{5}{2} = 2.5$.

In concluding, the student difficulties were grater during the third iteration and lower during the third iteration of the ACE cycle. This seems to be logical due to the increasing difficulty of the topics tackled in each of the tree iterations of the ACE teaching approach.

5. Conclusion

The MC model developed in the present work for studying the ACE teaching style, i.e. the pedagogical outcome of the APOS instructional treatment of mathematics, led to a numerical measure of the student difficulties during the several iterations of the ACE cycle for learning a certain mathematical topic. This is very useful for the mathematics instructor, because it helps the improvement of the corresponding APOS GD and the instructor's teaching plans in general for enhancing the student performance. The classroom application presented on teaching the graphical representation of the derivative illustrated the applicability and usefulness of our model in practice.

Our plans for future research on the subject include the effort of combining MCs with other suitable mathematical tools like fuzzy logic, grey system theory, etc., for improving the effectiveness of our methods for representing mathematically the APOS/ACE instructional treatment of mathematics. On the other hand, our second target focuses on more applications of similar methods to other mathematical topics (e.g. see [18].

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