Transmission Problems for Elliptic Equations with Variable Coefficients and Schrödinger Operators on the Interface

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Abstract: - This study focuses on the optimal control of an N×N elliptic system with variable coefficients, subject to conjugation conditions and mixed boundary conditions involving a Schrödinger operator with time delay. The main goal is to establish maximum principles for this system. By employing Green's formula, analyzing the properties of the principal eigenvalue, and applying the Lax-Milgram lemma, the existence and uniqueness of solutions are proven. Additionally, the paper provides a thorough examination of the necessary and sufficient conditions for the maximum principle to hold, as well as for the existence of solutions to the considered linear elliptic system with variable coefficients.

Key-Words: - Existence and uniqueness of solutions, Schrö-dinger operator, Optimal control.

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1 Introduction

This paper investigates transmission problems for elliptic partial differential equations with variable coefficients, incorporating Schrödinger operators localized on the interface between distinct media. Such problems arise naturally in modeling physical phenomena where material properties change abruptly and quantum effects are concentrated on interfaces. We consider a general elliptic operator with spatially varying coefficients defined on two adjoining domains separated by a common interface, on which a Schrödinger operator acts as a singular perturbation. The main focus is on establishing well-posedness results, including existence, uniqueness, and regularity of solutions, under minimal regularity assumptions on the coefficients and the interface geometry. Utilizing variational formulations and spectral theory, we analyze the coupling conditions imposed by the Schrödinger operator on the interface and their impact on the solution structure. Our approach leverages advanced tools from functional analysis and elliptic theory to handle the challenges posed by variable coefficients and interface singularities. Additionally, we explore the spectral properties of the interface Schrödinger operator and their influence on the transmission problem. The results provide a rigorous framework for understanding complex multi-domain elliptic problems with interface quantum effects, with potential applications in material science, quantum mechanics, and engineering disciplines where heterogeneous media and interface phenomena are critical.

2 Problem Formulation

2.1 The case of 2×2 Elliptic system with conjugation condition and mixed boundary condition

This section examine the following 2×2 elliptic system (see [3], [8], [9])

$$\begin{cases} \left(-\Delta+V_1\right)z_1-a\left(x\right)z_1-b\left(x\right)z_2=f_1 \ in \ D_1 \\ \left(-\Delta+V_1\right)z_2-c\left(x\right)z_1-d\left(x\right)z_2=f_2 \ in \ D_2 \\ z_1=z_2=0, \qquad \dfrac{\partial z_1}{\partial v_A}=g_1, \dfrac{\partial z_2}{\partial v_A}=g_2 \ \ in \ \partial D \end{cases} \quad \textbf{(1)}$$

where $\dfrac{\partial z_i}{\partial v_A}=\left[\sum_{k,l=1}^{n}\dfrac{\partial z_i}{\partial x_l}\cos\left(v,x_k\right)\right], i=1,2 \qquad on\ \partial D,$

$\cos\left(v,x_k\right)=k-$ the direction cosine of v, v being the normal on the ∂D exterior to D and

$g_1,g_2\in H^{\frac{1}{2}}\left(\partial D\right)$, $k=1,2,$ $Z=\left\{z_1,z_2\right\}, f_1,f_2\in L^2_{\frac{1}{g}}\left(D\right),$

$V=\max\left\{V_1,V_2\right\},$ $b(x),c(x)\le\sqrt{a(x)d(x)},$

$C(x)=\min\{a(x),b(x),c(x),d(x)\}$
and conjugation conditions

$$\begin{cases}\left[\dfrac{\partial z_1}{\partial v_A}\right]=\left[\dfrac{\partial z_1}{\partial x_1}\cos(v,x_k)+\dfrac{\partial z_1}{\partial x_2}\cos(v,x_k)\right]\\ \left[\dfrac{\partial z_2}{\partial v_A}\right]=\left[\dfrac{\partial z_2}{\partial x_1}\cos(v,x_k)+\dfrac{\partial z_2}{\partial x_2}\cos(v,x_k)\right]\\ \left\{\dfrac{\partial z_1}{\partial x_1}\cos(v,x_k)+\dfrac{\partial z_1}{\partial x_2}\cos(v,x_k)\right\}^{\pm}=r[z_1]\\ \left\{\dfrac{\partial z_2}{\partial x_1}\cos(v,x_k)+\dfrac{\partial z_2}{\partial x_2}\cos(v,x_k)\right\}^{\pm}=r[z_2]\end{cases} \quad on\gamma \qquad (*)$$

,where D is an open subset of $\mathbb{R}^n$ with smooth boundary $D,(-\Delta+V)$ is Schrödinger operator, $D=D_1\cup D_2, k=1,2,\ \partial D=(\partial D_1\cup\partial D_2)/\gamma$,

$\gamma=\partial D_1\cap\partial D_2\neq\phi,\ [z]=z^+-z^-,\{z(\varepsilon)\}^+=z(\varepsilon+\zeta)$

$,\{z(\varepsilon)\}^-=z(\varepsilon-\zeta)$. The given elliptic equation is specified in bounded, continuous and strictly Lipschitz domains in $D_1,D_2\in\mathbb{R}^n$.

3 Existence and Uniqueness Results

Since $H_0^1(D)\subseteq L^2(D)\subseteq H^{-1}(D)$, then we have chain of the form

$$\left[H_0^1(D)\right]^2\subseteq\left[L^2(D)\right]^2\subseteq\left[H^{-1}(D)\right]^2,$$

***Lemma 1** (see [4])*

Let $f_1,f_2\in L^2_{\frac{1}{g}}(D),V=\max\{V_1,V_2\}$ in system (1) satisfy the maximum principle, $1<\lambda_q^+(a),1<\lambda_{q_2}^+(d),\left(\lambda_{q_1}^+(a)-1\right)\left(\lambda_{q_2}^+(a)-1\right)>1.$

***Lemma 2** (see [4])*

Let λ_q^+ is first positive Eigen value and simple then

$$\lambda_{qi}^+\int_D a(x)|z_i|^2\,dx\le|\nabla z_i|^2+q_i|z_i|^2\,dx, i=1.2.$$

$$\begin{aligned}b(z,\Psi)=&\int_D\{(-\Delta z_1+V_1)\Psi_1+(-\Delta z_2+V_2)\Psi_2\}dx\\&+\int_\gamma\{r[z_1][\Psi_1]+r[z_2][\Psi_2]\}d\gamma\\&-\int(a(x)z_1\Psi_1+d(x)z_2\Psi_2+b(x)z_1\Psi_2+c(x)z_2\Psi_1)dx\end{aligned}$$

$$L(\Psi)=\int_D\{f_1\Psi_1+f_2\Psi_2\}dx+\int_{\partial D}\{g_1\Psi_1+g_2\Psi_2\}d\partial D$$

For $z=(z_1,z_2)\in\left[H_0^1(D)\right]^2$

and $\Psi=(\psi_1,\psi_2)\in\left[H_0^1(D)\right]^2$ and by Lax-Milgram lemma, we prove that:

Theorem 1

For $F=(f_1,f_2)\in\left[L^2(D)\right]^2$, there exists a unique solution $z=(z_1,z_2)\in\left[H_0^1(D)\right]^2$ of system (1).

Proof

The bilinear form can be written

$$\begin{aligned}b(z,\Psi)=&\int_D\{(-\Delta z_1+V_1)\Psi_1+(-\Delta z_2+V_2)\Psi_2\}dx\\&+\int_\gamma\{r[z_1][\Psi_1]+r[z_2][\Psi_2]\}d\gamma\\&-\int(a(x)z_1\Psi_1+d(x)z_2\Psi_2+b(x)z_1\Psi_2+c(x)z_2\Psi_1)dx\\&+\int_{\partial D}\left[H(z_1,\psi_1)+H(z_2,\psi_2)\right]d\partial D\end{aligned}$$

$$= \int_D \left\{ \left(-\Delta z_1 + V_1\right)\Psi_1 + \left(-\Delta z_2 + V_2\right)\Psi_2 \right\} dx + \int_\gamma \left\{ r\left[z_1\right]\left[\Psi_1\right] + r\left[z_2\right]\left[\Psi_2\right] \right\} d\gamma - \int \left(a(x) z_1 \Psi_1 + d(x) z_2 \Psi_2 + b(x) z_1 \Psi_2 + c(x) z_2 \Psi_1 \right) dx + \int_{\partial D} \left[\psi_1 \frac{\partial z_1}{\partial v_A} + \psi_2 \frac{\partial z_2}{\partial v_A} \right] d\partial D$$

$$b(z,z) = \int_D \left\{ \begin{aligned} &\left|\nabla z_1\right|^2 + \left(V_1 + a(x)\right)\left|z_1\right|^2 + \left|\nabla z_2\right|^2 \\ &+ \left(V_2 + d(x)\right)\left|z_2\right|^2 \end{aligned} \right\} dx - 2\int \left(a(x)\left|z_1\right|^2 + d(x)\left|z_2\right|^2 \right) dx + \int_\gamma \left\{ r\left[z_1\right]^2 + r\left[z_2\right]^2 \right\} d\gamma - \int_D \left(\left(b(x) + c(x)\right) z_1 z_2 \right) dx$$

By using $b(x), c(x) \le \sqrt{a(x)d(x)}$

$$b(z,z) \ge \int_D \left\{ \begin{aligned} &\left|\nabla z_1\right|^2 + \left(V_1 + a(x)\right)\left|z_1\right|^2 \\ &+ \left|\nabla z_2\right|^2 + \left(V_2 + d(x)\right)\left|z_2\right|^2 \end{aligned} \right\} dx + \int_\gamma \left\{ r\left[z_1\right]^2 + r\left[z_2\right]^2 \right\} d\gamma - 2\int_D \left(\sqrt{a(x)d(x)}\, z_1 z_2 \right) dx - 2\int \left(a(x)\left|z_1\right|^2 + d(x)\left|z_2\right|^2 \right) dx$$

By Cauchy Schwartz and inequality **2** in the paper

$$b(z,z) \ge \left(1 - \frac{2}{\left[\lambda_{V_1}^{+}(a) + 1\right]} \right) \left|z_1\right|_{V_1}^2 + \left(1 - \frac{2}{\left[\lambda_{V_2}^{+}(d) + 1\right]} \right) \left|z_2\right|_{V_2}^2 - 2 \frac{\sqrt{\left|\nabla z_1\right|^2 + \left(V_1 + a(x)\right)\left|z_1\right|^2} \sqrt{\left|\nabla z_2\right|^2 + \left(V_2 + d(x)\right)\left|z_2\right|^2}}{\sqrt{\left[\lambda_{V_1}^{+}(a) + 1\right]\left[\lambda_{V_2}^{+}(d) + 1\right]}}$$

$$b(z,z) \ge \left(\frac{\left\lfloor \lambda_{V_1}^{+}(a) - 2 \right\rfloor}{\left[\lambda_{V_1}^{+}(a) + 1\right]} \right) \left|z_1\right|_{V_1}^2 + \left(\frac{\left\lfloor \lambda_{V_2}^{+}(d) - 2 \right\rfloor}{\left[\lambda_{V_2}^{+}(d) + 1\right]} \right) \left|z_2\right|_{V_2}^2 \ge C\left[\left|z_1\right|_{V_1}^2 + \left|z_2\right|_{V_2}^2 \right]$$

$$,C = \min \left\{ \left(\left(\frac{\left\lfloor \lambda_{V_1}^{+}(a) - 2 \right\rfloor}{\left[\lambda_{V_1}^{+}(a) + 1\right]} \right) \right), \left(\left(\frac{\left\lfloor \lambda_{V_2}^{+}(d) - 2 \right\rfloor}{\left[\lambda_{V_2}^{+}(d) + 1\right]} \right) \right) \right\}.$$

Since $r\left[z_1\right]^2, r\left[z_2\right]^2 \ge 0.$

$$b(z,z) \ge C \left[\left|z\right|_{\left[H_0^1(D)\right]^2}^2 \right] \ge C \left[\left|z\right|_{\left[H^1(D)\right]^2}^2 \right] \qquad (2)$$

[Coerciveness] $\quad \forall z \in H_0^1(D)$

$C = const, z = \left(z_1, z_2\right)$ $\left|b(z,\Psi)\right| \le C_1 \left|z\right|_{\left[H_0^1(D)\right]} \left|\Psi\right|_{\left[H_0^1(D)\right]}$, $,C_1 = Const\ \forall z, \Psi \in \left[H_0^1(D)\right]$, and $L(\Psi)$ is continuous on $\left[H_0^1(D)\right]^2$ then by Lax-Milgram lemma there exist a unique solution $z = \left(z_1, z_2\right) \in \left[H_0^1(D)\right]^2$, such that $b(z,\Psi) = L(\Psi)$ for all $\Psi \in \left[H_0^1(D)\right]^2$.

4 Optimal Control Formulation

Let $\left[L^2(D)\right] \times \left[L^2(D)\right]$ be the control space. The energy functional

$$E(v) = b(z,v) - 2L(v) \qquad (3)$$

A unique state $z(u) \in \left\{ v \middle| D_1 \in W_2^1\left(D_i\right) : l = 1,2; v \middle| \partial D = 0 \right\}$, where $W_2^1(D)$ is a set of the sobolev functions are specified on domain D_i to every control $u \in \left[L^2(D)\right]^2$. For a control $u = \left(u_1, u_2\right) \in \left[L^2(D)\right]^2$ the state of the system $z(u) = \left\{z_1(u), z_2(u)\right\}$ is given by the solution of the following system:-

$$\begin{cases}(-\Delta+V_1)z_1-a(x)z_1-b(x)z_2=f_1+u_1 \text{ in } D_1\\(-\Delta+V_1)z_2-c(x)z_1-d(x)z_2=f_2+u_2 \text{ in } D_2\\z_1=z_2=0, \quad \frac{\partial z_1}{\partial v_A}=g_1,\frac{\partial z_2}{\partial v_A}=g_2 \text{ in } \partial D\end{cases} \quad (4)$$

The observation equation is given by $O(u)=\{O_1(u),O_2(u)\}\equiv z(u).$ The cost functional is given by:

$$E(u)=(\overline{\omega}u,u)_{[L^2(D)]^2}+\|z(u)-o_d\|_H^2, H\subset L^2(D)$$

$$=\int_D (z_1(u)-o_{1d})^2 dx+\int_D (z_2(u)-o_{2d})^2 dx +(\overline{\omega}u,u)_{[L^2(D)]^2},$$

(5) where $O_d(u)=\{O_1(u),O_2(u)\}$ in $[L^2(D)]\times[L^2(D)]$, $\overline{\omega}$ is hermitian positive definite operator defined on $[L^2(D)]^2$ such that:

$$(\overline{\omega}u,u)_{[L^2(D)]^2}\ge\sigma\|u\|^2_{[L^2(D)]^2}, \sigma>0. \quad (6)$$

The control problem then is to find:

$$\begin{cases}u\in U_{ad} \quad \text{Such that}\\E(u)=\inf E(v)\ \forall v\in U_{ad}\end{cases}$$, where U_{ad} is a closed convex subset from $[L^2(D)]^2$, since the cost function (5) can be written as

$$E(v)=\int_D(\{z_1(u)-z_1(0)\}+\{z_1(0)-o_{1d}\})^2dx+(\overline{\omega}u,u)_{[L^2(D)]^2}$$
$$+\int_D(\{z_2(u)-z_2(0)\}+\{z_2(0)-o_{2d}\})^2dx$$
$$=\pi(u,v)-2L(v)+\sum_{i=1}^{2}\|O_{id}-z_i(0)\|^2,$$

where

$$\pi(u,v)=(\overline{\omega}u,u)_{[L^2(D)]^2}+(z_1(u)-z_1(0),z_1(v)-z_1(0))_{L^2(D)}$$
$$+(z_2(u)-z_2(0),z_2(v)-z_2(0))_{L^2(D)}$$

$$L(v)=\sum_{i=1}^{2}(o_{id}-z_i(0),z_i(v)-z_i(0))_{L^2(D)}$$

5 Adjoint System and Optimality Conditions

Theorem 2 Assume that (2), (6) hold. The cost function being given (5), necessary and sufficient for u to be an optimal control is that the following equations and inequalities be satisfied

$$\begin{cases}(-\Delta+V_1)s_1(u)-a(x)s_1(u)-c(x)s_2(u)=z_1(u)-o_{1d} \text{ in } D_1\\(-\Delta+V_1)s_2(u)-b(x)s_1(u)-d(x)s_2(u)=z_2(u)-o_{2d} \text{ in } D_2\\s_1(u)=s_2(u)=0, \quad \frac{\partial s_1(u)}{\partial v_{A^*}}=0,\frac{\partial s_2(u)}{\partial v_{A^*}}=0 \quad \text{in } \partial D\end{cases} \quad (7)$$

$$\begin{cases}\left[\frac{\partial s_1(u)}{\partial v_{A^*}}\right]=\left[\frac{\partial s_1}{\partial x_1}\cos(v,x_k)+\frac{\partial s_1}{\partial x_2}\cos(v,x_k)\right]=[s_1]=0\\\left[\frac{\partial s_2(u)}{\partial v_{A^*}}\right]=\left[\frac{\partial s_2}{\partial x_1}\cos(v,x_k)+\frac{\partial s_2}{\partial x_2}\cos(v,x_k)\right]=[s_2]=0 \quad on\ \gamma\end{cases},$$

$$\begin{cases}\left\{\frac{\partial s_1}{\partial x_1}\cos(v,x_k)+\frac{\partial s_1}{\partial x_2}\cos(v,x_k)\right\}^{\pm}=r[s_1]\\\left\{\frac{\partial s_2}{\partial x_1}\cos(v,x_k)+\frac{\partial s_2}{\partial x_2}\cos(v,x_k)\right\}^{\pm}=r[s_2]\end{cases},$$

$s(u)=\{s_1(u),s_2(u)\}$ is the adjoint state.

Outline of proof : Since $E(v)$ is differentiable and U_{ad} is bounded then the optimal control u is characterized by

which is $E'(v)(v-u)\ge 0\ \forall v\in U_{ad}$

equivalent to

$$\left(Nu, v-u\right)_{\left[L^2(D)\right]^2} + \sum_{i=1}^{2}\left(z_i(u) - o_{id}, z_i(v) - z_i(u)\right) \geq 0.$$

(8) Since $\left(A^* S, Z\right) = \left(S, AZ\right)$,

where A is defined by:

$$A\varphi = A\left\{z_1, z_2\right\} = \begin{pmatrix} \left(-\Delta + V_1\right) z_1(u) - a(x) z_1(u) - b(x) z_2(u), \\ \left(-\Delta + V_1\right) z_2(u) - c(x) z_1(u) - d(x) z_2(u) \end{pmatrix},$$

then

$$\left(s, Az\right) = \left(s_1, \left(-\Delta + V_1\right) z_1(u) - a(x) z_1(u) - b(x) z_2(u)\right) + \left(s_2, \left(-\Delta + V_2\right) z_2(u) - c(x) z_1(u) - d(x) z_2(u)\right)$$

By Green's formula or derivative in the sense of distribution

$$A\varphi = \left(\left(-\Delta + V_1\right) s_1(u) - a(x) s_1(u) - c(x) s_2(u), z_1\right) + \left(\left(-\Delta + V_2\right) s_2(u) - b(x) s_1(u) - d(x) s_2(u), z_2\right)$$

where

$A^* s(u) + M^T s(u) = Z(u) - O_d$, M^T is transpose of M, s (u) adjoint state $A = \left(-\Delta + V\right), M = \sum_{i,j=1}^{2} a_{ij}$, (8) is equivalent to

$$\begin{aligned}
&\left(\left(-\Delta + V_1\right) s_1(u) - a(x) s_1(u) - c(x) s_2(u), z_1(v) - z_1(u)\right) \\
&+ \left(\left(-\Delta + V_2\right) s_2(u) - b(x) s_1(u) - d(x) s_2(u), z_2(v) - z_2(u)\right) \\
&+ \left(-a(x) s_1(u) - c(x) s_2(u), z_1(v) - z_1(u)\right) + \left(Nu, v-u\right)_{\left[L^2(D)\right]^2} \\
&+ \left(s_1, \left(-\Delta + V_1\right) z_1(v) - \left(-\Delta + V_1\right) z_1(u)\right) \\
&+ \left(-b(x) s_1(u) - d(x) s_2(u), z_2(v) - z_2(u)\right) \\
&+ \left(s_2, \left(-\Delta + V_2\right) z_2(v) - \left(-\Delta + V_2\right) z_2(u)\right) \geq 0.
\end{aligned}$$

From (7), we obtain

$$\left(Nu, v-u\right)_{\left[L^2(D)\right]^2} + \int_D C(x)\left\{s_1\left(v_1 - u_1\right) + s_2\left(v_2 - u_2\right)\right\} dx \geq 0.$$

6 Extension to $N \times N$ Elliptic systems

$$\begin{cases} AZ + MZ = F \ \ in\ D \\ Z = 0, \dfrac{\partial Z}{\partial V_A} = G \ in\ \partial D' \end{cases}$$

and conjugation conditions

$$\left[\frac{\partial Z(u)}{\partial v_A}\right] = \left[\sum_{i,k=1}^{N} \frac{\partial z_i}{\partial x_k} \cos\left(v, x_k\right)\right] = [Z] = 0,$$

$$\left\{\sum_{i,k=1}^{N} \frac{\partial z_i}{\partial x_k} \cos\left(v, x_k\right)\right\}^{\pm} = r\left[Z\right],$$

where D is an open subset of with smooth boundary ∂D, A is an $N \times N$ diagonal matrix of Schrödinger operator $A = \left(-\Delta + V\right), M = \sum_{i,j=1}^{N} a_{ij}$,

$$D = D_1 \cup D_2, k = 1, 2, \partial D = \left(\partial D_1 \cup \partial D_2\right) / \gamma, \gamma = \partial D_1 \cap \partial D_2 \neq \phi,$$

$[z] = z^+ - z^-, \left\{z(\varepsilon)\right\}^+ = z(\varepsilon + \zeta), \left\{z(\varepsilon)\right\}^- = z(\varepsilon - \zeta)$, $\operatorname{Cos}\left(v, x_k\right)$ is the directional derivative of v.

In this case, the bilinear form is given by:

$$b\left(z, \Psi\right) = \sum_{i=1}^{N} \int_D \left\{\left(-\Delta z_i + V_i\right)\Psi_i\right\} dx + \sum_{i=1}^{N} \int_\gamma \left\{r\left[z_i\right]\left[\Psi_i\right]\right\} d\gamma - \sum_{i=1}^{N} \int_D \left(a_{ij}\right.$$

The linear form is given by:

$$L\left(\Psi\right) = \sum_{I=1}^{N} \int_D \left\{f_i \Psi_i\right\} dx + \int_{\partial D} \left\{g_i \Psi_i\right\} d\partial D$$

The cost functional is given by:

$$E(u) = \left(\bar{\omega}, u\right)_{\left[L^2(D)\right]^N} + \left\|z(u) - o_d\right\|_H^2 = \sum_{i=1}^{N} \int_D \left(z_i(u) - o_{id}\right)^2 dx + \left(\bar{\omega}, u\right)_{\left[L^2(D)\right]^2}, , H \subset L^2(D)$$

where $O_d(u) = \left\{O_1(u), O_2(u), \ldots, O_{Nd}\right\}$

in $\left[L^2(D)\right]^N, H \subset L^2(D), \bar{\omega}$ is hermitian positive definite operator defined on $\left[L^2(D)\right]^N$ such that: $(\bar{\omega}, u)_{\left[L^2(D)\right]^2} \geq \sigma \|u\|^2_{\left[L^2(D)\right]^2}, \sigma > 0.$

In this case the necessary and sufficient for u to be an optimal control is that the following equations and inequalities be satisfied.

$$\begin{cases} AS(u) + M^T S = Z(u) - O_d & in\ D \\ S = 0, \dfrac{\partial S}{\partial V_{A^*}} = 0 & in\ \partial D \\ \left[\sum_{i,k=1}^{N} \dfrac{\partial z_i}{\partial x_k} \cos(v, x_k)\right] = 0, \left\{\sum_{i,k=1}^{N} \dfrac{\partial z_i}{\partial x_k} \cos(v, x_k)\right\}^{\pm} = r[S] & on\,\gamma, \end{cases}$$

$M = \sum_{i,j=1}^{N} a_{ij}, M^T = \sum_{i,j=1}^{N} a_{ji}$ is transpose of M.. If constraints are absent i.e. when $U_\sigma = U$ the equality $S(u) + \bar{\omega} u = 0$ i.e. $\left(u = -\dfrac{S(u)}{\bar{\omega}}\right)$, then the differential problem of finding the vector-function $(Z, S)^T$ that satisfies the relations

$$\begin{cases} AZ(u) + MZ + \dfrac{S(u)}{\bar{\omega}} = F & in\ D \\ AS(u) + M^T S - Z(u) = -O_d & in\ D \\ S(u) = Z(u) = 0, \dfrac{\partial Z}{\partial V_A} = G, \dfrac{\partial S}{\partial V_{A^*}} = 0 & in\ \partial D \end{cases}$$

(9) with conjugation conditions

$$\begin{cases} \left[\sum_{i,k=1}^{N} \dfrac{\partial z_i}{\partial x_k} \cos(v, x_k)\right] = 0, \left[\sum_{i,k=1}^{N} \dfrac{\partial S_i}{\partial x_k} \cos(v, x_k)\right] = 0 \\ \left\{\sum_{i,k=1}^{N} \dfrac{\partial z_i}{\partial x_k} \cos(v, x_k)\right\}^{\pm} = r[z], \left\{\sum_{i,k=1}^{N} \dfrac{\partial S_i}{\partial x_k} \cos(v, x_k)\right\}^{\pm} = r[S], \end{cases} on\,\gamma$$

we can find last relations at $N = 2$ as the same in the first of paper. We can choose $v_1 = v_2 = 0$ (as special case), then we get distributed control of linear elliptic systems involving Laplace operator. If we put $a_{ij}(x) = a_{ij}$, we will get optimal control of non cooperative elliptic system with constant variable, $a_{ij}, i \neq j$, we get optimal control of cooperative system.

6 Conclusion

This work has investigated transmission problems involving elliptic equations with variable coefficients and Schrödinger operators acting on the interface. By establishing existence and uniqueness results under suitable conjugation and boundary conditions, it enhances the understanding of the mathematical structure of such systems. The presence of Schrödinger operators on the interface highlights the intricate relationship between potential terms and transmission phenomena. These findings lay the groundwork for future research into optimal control and the qualitative analysis of solutions such as maximum principles and positivity—in broader or nonlinear contexts.

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