Examining a Mixed Inverse Approach for Stable Control of a Rover

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Abstract: In this paper we significantly extend previous work on methods for ensuring that control system behavior is invariant with respect to units chosen for critical state variables when intermediate operations require solutions to underdetermined or overdetermined systems of equations. For example, least-squares methods are intrinsically sensitive to whether lengths are defined in units of, e.g., centimeters or meters. Prior work has argued that many practical control systems have such unrecognized unit dependencies and are thus vulnerable to exhibiting unexpected behaviors in some situations. Here we extend the underlying theory of unit-consistent generalized inverses (UC inverse) to the more common practical situation in which some state variables have unit dependencies while others require consistency with respect to rigid rotations. We also extend the theory of UC inverse by formally proving that their consistency guarantees are preserved under Kronecker (tensor) products, which is a critical property for using and analyzing complex control systems defined as compositions of simpler subsystems.

Key–Words: Control Systems, Generalized Matrix Inverse, Inverse Problems, Kronecker product, Linear Estimation, Linear Systems, Moore-Penrose Pseudoinverse, System Design, UC Generalized Inverse, Unit Consistency.

1 Introduction

There has been a renewed recognition of the importance of preserving critical consistency conditions when designing control systems, e.g., one should expect the behavior of a system to be the same regardless of whether its length or distance variables are defined consistently in units of meters or centimeters. This is important because if the behavior is sensitive to the choice of units then its reliability can only be assessed with respect to a specific scenario or set of operating assumptions. For example, depending on the formulation of the system it may exhibit one behavior when initialized at a 2-dimensional location $(0, 0)$ and a very different behavior if initialized at location $(1000, 1000)$ when clearly the choice of coordinate frame and origin should be entirely irrelevant to the system's performance. It is a fundamental design principle that sensitivity to arbitrary application-specific details should be minimized whenever possible. In general, if a system is defined and performs well in some particular Euclidean coordinate frame then it should be expected to perform identically if that coordinate frame is arbitrarily rotated or scaled. Similarly, the performance of the system should not be affected if its key parameters are all consistently defined in metric units or in imperial units.

The Moore-Penrose pseudoinverse [1,2] (MP inverse), A^{-P}, is defined for any $m \times n$ matrix A and satisfies the fundamental generalized inverse properties as well as the following for any conformant unitary/orthonormal matrices U and V:

$$(UAV)^{-P} = V^* A^{-P} U^* \tag{1}$$

The MP inverse is applicable to problems defined in a Euclidean state space for which the behavior of the system of interest should be invariant with respect to arbitrary rotations of the coordinate frame.

In other contexts consistency must be preserved with respect to changes of *units*, e.g., from imperial to metric or from meters to kilometers, rather than with respect to rotations of a global coordinate frame. For example, in the case of two state variables representing voltage and pressure, respectively, no physical meaning or interpretation can be given to a "rotation" of this 2-dimensional subspace to one in which units of voltage and pressure are mixed. In other words, the relative effect of a rigid rotation applied to the two variables implicitly assumes they are defined in the same units, when of course the choice of units for voltage and pressure are fundamentally incommensurate. This means the result of the rotation will be strongly sensitive to the arbitrary choice of units on the two variables in a way that has no meaning with respect to the application of interest.

This kind of unit consistency requires a generalized inverse A^{-U} that satisfies

$$(DAE)^{-U} = E^{-1} A^{-U} D^{-1} \tag{2}$$

where the diagonal matrix D represents units on variables in one space and the diagonal matrix E represents different units for the same variables in a different space. The MP

Inverse does *not* satisfy Eq. (2):

$$(DAE)^{-P} \;=\; E^{-1}A^{-P}D^{-1} \tag{3}$$

which implies that the MP inverse should not be applied to systems in which unit consistency must be preserved. More generally, this implies that operations involving the minimization of a squared-error (i.e., rotation-invariant ℓ_2-norm) criterion should not be applied when unit consistency must be preserved. As will be discussed, this largely unrecognized fact means that many practical engineering systems suffer from unrecognized sensitivities to the choice of units on critical state variables. Such sensitivities not only can lead to suboptimal performance but also to unexpected instabilities and potentially serious failure modes.

In previous works [3, 4] it was shown that the UC inverse A^{-P} developed in [5, 6] guarantees controls that are unit invariant. In this paper we consider the more general problem in which the kinematics of a robotic system include a subset of variables that require unit consistency while others require consistency with respect to rigid rotations. Specifically, we demonstrate in the case of a rover that preserving the appropriate consistency conditions for all state variables ensures behavior that is invariant to the choice of units on length variables and with respect to rotations of sets of variables that are jointly defined within a Cartesian coordinate frame. Although we demonstrate the effectiveness of the approach with these two specific consistency conditions, the principal conclusion to be drawn is that the approach does in fact generalize as theory predicts to ensure invariant system behavior with respect to an arbitrary mix of consistency conditions assumed across subsets of the system's state variables.

After demonstrating an application of the general approach to the rover problem, we proceed to remedy a current deficiency in the theory of the UC inverse by formally proving that its properties are preserved under Kronecker (tensor) products.

2 Generalized Consistency Considerations for Multi-dimension Problem

In this section we examine how the UC inverse can be combined with the MP inverse, and even other generalized inverses (e.g., the Drazin inverse [7–9] or other similarity-consistent inverse [10]), to construct solutions to inverse problems when there is a mix of variables involving different consistency requirements. In addition to variables that require unit consistency to be preserved, a complex real-world system may also involve variables defined in a Cartesian coordinate frame that require consistency with respect to rotations of that coordinate frame. In other words, the behavior of the control system must be invariant with respect to changes of units for some variables and invariant with respect to rotations for other variables. The UC inverse is applicable in one case while the MP inverse is applicable in the other, but what is needed for such a system is a generalized inverse that will guarantee unit consistency for some variables and rotation consistency for others. If we assume[1] that the first m variables require unit consistency, and the remaining n variables require rotation consistency, then the transformation matrix to be inverted can be block-partitioned as

$$A \;=\; \underbrace{\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}}_{m \;\; n} \begin{matrix} \}\, m \\ \}\, n \end{matrix} \tag{4}$$

It has been shown [6] that the mixed inverse can be obtained from this block-partitioned form as

$$A^{-M} = \begin{bmatrix} (W - XZ^{\text{-P}}Y)^{\text{-U}} & -W^{\text{-U}}X(Z - YW^{\text{-U}}X)^{\text{-P}} \\ -Z^{\text{-P}}Y(W - XZ^{\text{-P}}Y)^{\text{-U}} & (Z - YW^{\text{-U}}X)^{\text{-P}} \end{bmatrix} \tag{5}$$

Consider the control of a planetary rover with a robotic arm as displayed in Figure 1, which includes two projected views in directions D_1 and D_2. The x-y coordinate is shown as frame F. The frame F', which is an orthogonal transpose of frame F by θ', will be considered later. The rover is free to move in any direction on planar terrain, and its Cartesian position coordinates in this plane are (x_1, y_1). The part B can rotate and can also ascend/descend[2]. In addition, the arm can elongate within a fixed range, but it cannot rotate in the vertical plane. Thus there are 5 degrees of freedom for the design, denoted as $q = [\theta_1, l, x_1, y_1, z_1]^T$, where (x_1, y_1, z_1) is the position coordinates of part B.

From the geometry relations, the position of tip-point P_A is given as $P_A = [x, y, z, 0, 0]^T$,

$$x \;=\; x_1 + l \cdot \sin\theta_0 \cdot \cos\theta_1 \tag{6}$$

$$y \;=\; y_1 + l \cdot \sin\theta_0 \cdot \sin\theta_1 \tag{7}$$

$$z \;=\; z_1 - l \cdot \cos\theta_0 \tag{8}$$

We can then generate the Jacobian matrix for $\vec{v} = J\dot{q}$ as

$$J = \begin{bmatrix} -l \cdot \sin\theta_0 \sin\theta_1 & \sin\theta_0 \cos\theta_1 & 1 & 0 & 0 \\ l \cdot \sin\theta_0 \cos\theta_1 & \sin\theta_0 \sin\theta_1 & 0 & 1 & 0 \\ 0 & -\cos\theta_0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The initial states are set to be $\theta_1 = 45^\circ$ and $l = 1.0m$, the constant angle $\theta_0 = 45^\circ$. The target velocity of the

[1] The ordering of the variables is arbitrary so there is no loss of generality in assuming they are permuted so that the UC variables come first and the rotation-consistent variables come next.

[2] The part B can be thought of as an extendible arm for taking a soil or rock sample.

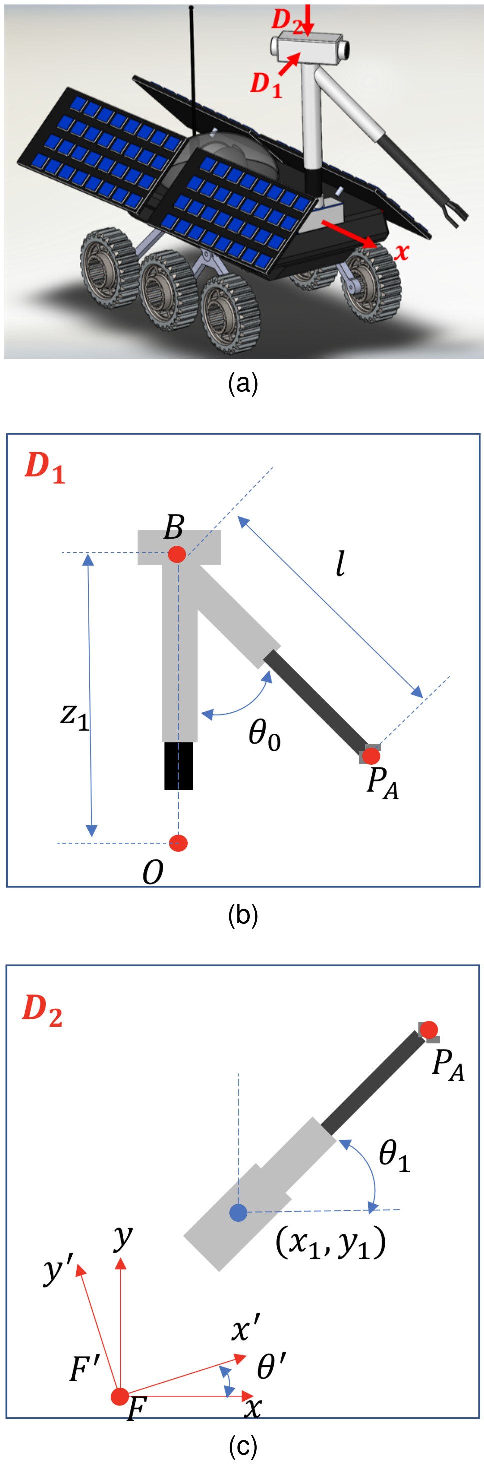

Figure 1: Rover with an extendable arm. The rover is free to move on a plane, body part B can ascend/descend or rotate, and the arm can be extended. The two projected views, D_1 and D_2, show the structure of the arm. $\theta_0 = 45°$ is a fixed angle. The coordinate frame F' is a rotation of the original frame F by an angle of θ'.

tip-point is $\vec{v} = [2, 0, -1, 0, 0]m/s$. Since the system has redundant degrees of freedom and therefore J is singular, $\dot{q} = J^{\widetilde{-1}}\vec{v}$ has to be solved with a general inverse (e.g. MP inverse, UC inverse, or mixed inverse). For the unknown variables in $\dot{q}$, $[\dot{\theta}_1, \dot{l}]$ have incommensurate units, and $[\dot{x}_1, \dot{y}_1, \dot{z}_1]$ are defined in a common Euclidean space. Thus the Jacobian matrix can be partitioned as

$$W = \begin{bmatrix} -l \cdot \sin\theta_0 \sin\theta_1 & \sin\theta_0 \cos\theta_1 \\ l \cdot \sin\theta_0 \cos\theta_1 & \sin\theta_0 \sin\theta_1 \end{bmatrix}, X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -\cos\theta_0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the mixed inverse can be used to find the solution for $\dot{q}$ using equation 5. We initially use meters as the length unit and F as the coordinate frame. We will then consider a change of length units from meters to centimeters and a coordinate frame rotation from F to F' by a rotation angle of $\theta' = 30°$. Given $c = 100$ as the scale factor to convert from meters to centimeters, the governing equation for the centimeter and rotated case, $\vec{v}_{cm,F'} = J_{cm,F'}\dot{q}_{cm,F'}$, can be expressed as a diagonal transformation of the meter case, $\vec{v}_{m,F} = J_{m,F}\dot{q}_{m,F}$, as

$$\begin{bmatrix} c\cos(\theta'), & -c\sin(\theta'), & 0, & 0, & 0 \\ c\sin(\theta'), & c\cos(\theta'), & 0, & 0, & 0 \\ 0, & 0, & c, & 0, & 0 \\ 0, & 0, & 0, & c, & 0 \\ 0, & 0, & 0, & 0, & c \end{bmatrix} \cdot \vec{v}_{m,F}$$

$$= \begin{bmatrix} \cos(\theta'), & -\sin(\theta'), & 0, & 0, & 0 \\ \sin(\theta'), & \cos(\theta'), & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{bmatrix} \cdot J_{m,F}$$

$$\cdot \begin{bmatrix} c, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{bmatrix} \cdot \dot{q}_{cm,F'}$$

This shows the block of the matrix requiring rotation consistency and the block of variables defined in incommensurate units. We then test the three possible approaches to computing the controls: using the MP inverse alone; using the UC inverse alone; and using the mixed inverse obtained from equation 5. The solutions for $\dot{q}$ from the three approaches are displayed in table 1 for $t = 0s$. The column headings give the unit/coordinate frame in which each test was performed but the results are all given in (converted to) a common coordinate frame for comparison purposes. As can be seen, the mixed inverse is the only approach that produces identical results regardless of coordinate-system changes. For the other approaches, it can be seen that when the controls are evaluated with the MP inverse, it generates the same results when different coordinate frames are

Table 1: Inverse results. Joint velocities $\dot{q} = [\dot{\theta}_1(rad/s), \dot{l}(m/s), \dot{x}_1(m/s), \dot{y}_1(m/s), \dot{z}_1(m/s)]^T$, solved with MP inverse, UC inverse, and Mixed inverse approaches.

	Length unit(m) Coordinate(F)	Length unit(cm) Coordinate(F)	Length unit(cm) Coordinate(F')
MP inv	$\dot{q} = \begin{bmatrix} -0.6854 \\ 0.8536 \\ 1.1963 \\ -0.0498 \\ -0.3964 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.8179 \\ 0.8536 \\ 0.5734 \\ -0.5731 \\ -0.3964 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.8179 \\ 0.8536 \\ 0.5734 \\ -0.5731 \\ -0.3964 \end{bmatrix}$
UC inv	$\dot{q} = \begin{bmatrix} -1.2121 \\ 1.3536 \\ 0.6566 \\ -0.0101 \\ -0.0429 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.2121 \\ 1.3536 \\ 0.6566 \\ -0.0101 \\ -0.0429 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.4545 \\ 1.5690 \\ 0.3676 \\ -0.1943 \\ -0.1095 \end{bmatrix}$
Mixed inv	$\dot{q} = \begin{bmatrix} -1.8182 \\ 2.7071 \\ -0.3536 \\ -0.3536 \\ 0.9142 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.8182 \\ 2.7071 \\ -0.3536 \\ -0.3536 \\ 0.9142 \end{bmatrix}$	$\dot{q} = \begin{bmatrix} -1.8182 \\ 2.7071 \\ -0.3536 \\ -0.3536 \\ 0.9142 \end{bmatrix}$

used but not when length units are changed. When the UC inverse is solely applied to the entire system it generates invariant solutions when units are changed but not when rotations are applied. Therefore, it can be concluded that solely using either the UC inverse or MP inverse alone will not produce reliable results. Instead, the mixed inverse is required to ensure that the behavior of the system is invariant with respect to defined changes of units and coordinates.

A transient simulation was performed for $0.1s$ to further observe the full control process. Figure 2(a) displays variation of θ_1 for the three approaches. It shows that the angular velocity calculated over time by the MP inverse is not affected by a rotation of the coordinate frame from F to F' but is affected by a change of the length unit from meters to centimeters; and the reverse is true for the UC inverse. By contrast, the angular velocity from the mixed inverse is identical over time in all cases. In summary, for this system involving variables with different consistency requirements the mixed inverse yields reliable control while the alternatives do not. This demonstrates the necessity of using the appropriate inverse to satisfy all applicable consistency requirements. In the next section we proceed to formally prove that the approach demonstrated in this section can be rigorously applied to composite systems defined as tensor products of simpler subsystems.

3 UC Inverse and the Kronecker Product

The Kronecker product is often used for the mathematical representation of a complex system in terms of simpler subsystems [11]. In this section we show that the UC inverse

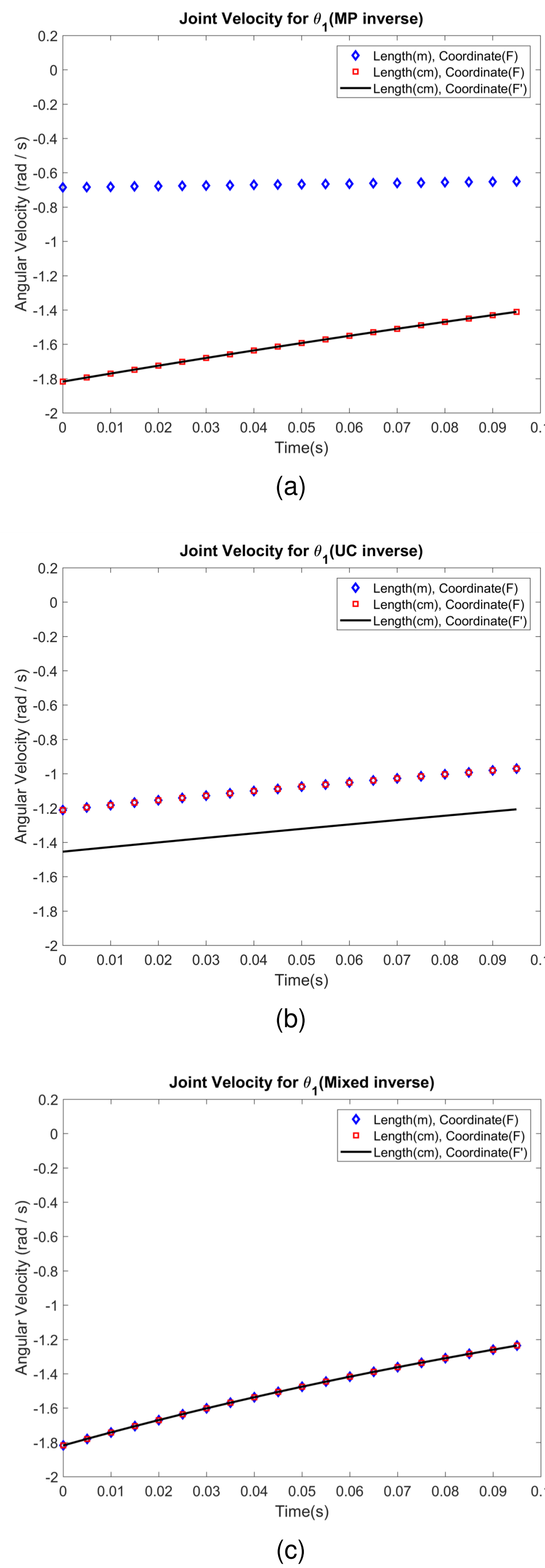

Figure 2: Joint velocity for θ_1 solved with different generalized inverse approaches. (a) MP inverse. (b) UC inverse. (c) mixed inverse. Only the mixed inverse yields the same results over the $0.1s$ for the transformations in all three cases.

satisfies the same useful properties as the MP inverse with respect to the Kronecker product.

The Kronecker product is a non-commutative tensor operator, usually denoted as $\otimes$, which takes an $m \times n$ matrix and a $p \times q$ matrix and constructs a composition of the two matrices to produce a higher-dimensional $mp \times nq$ matrix. The definition of the Kronecker product of $A_{m\times n}$ and $B_{p\times q}$ is

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{bmatrix}$$

Thus each $a_{i,j}B$ is a of $p\times q$ matrix. For example, assuming A is 2×2 and B is 3×2 (i.e., $m = n = 2$ and $p = 3, n = 2$) then

$$A \otimes B = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{1,1}b_{3,1} & a_{1,1}b_{3,2} & a_{1,2}b_{3,1} & a_{1,2}b_{3,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \\ a_{2,1}b_{3,1} & a_{2,1}b_{3,2} & a_{2,2}b_{3,1} & a_{2,2}b_{3,2} \end{bmatrix}$$

The Kronecker product is important in engineering design because it can be used to elegantly and efficiently represent complex systems as compositions of simpler subsystems. It finds applications in control systems, signal processing, image processing, semidefinite programming, and quantum computing [11–17]. It is bilinear and associative:

$$A \otimes (B + C) = A \otimes B + A \otimes C \tag{9}$$
$$(A + B) \otimes C = A \otimes C + B \otimes C \tag{10}$$
$$(kA) \otimes B = A \otimes (kB) = k(A \otimes B) \tag{11}$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \tag{12}$$

and it satisfies the following with respect to the transpose (and conjugate-transpose) operator

$$(A \otimes B)^T = A^T \otimes B^T \tag{13}$$

For matrices A, B, C and D for which the products AC and BD are valid, the following mixed-product property (so-called because it involves both standard matrix multiplication and the Kronecker product) can be shown to hold:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{14}$$

If matrices A and B are orthogonal then $A \otimes B$ is also orthogonal:

$$(A \otimes B)^T (A \otimes B) = I \tag{15}$$

It is also the case that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{16}$$

and more generally

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-1} = A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_n^{-1} \tag{17}$$

The last two properties are necessary to construct a matrix to transform from one Kronecker-constructed matrix to another Kronecker-constructed matrix, which is required for performing controls of the kinds of robotic and mechanical systems of interest in this paper, but we also require the ability to apply generalized matrix inverses in the case of singular matrices. It has been proven that the MP inverse satisfies [18]:

$$(A \otimes B)^{-P} = A^{-P} \otimes B^{-P} \tag{18}$$

and more generally:

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-P} = A_1^{-P} \otimes A_2^{-P} \otimes \cdots \otimes A_n^{-P} \tag{19}$$

but it has not yet been established that these two results also hold for the UC inverse. They must be proven in order to show that the UC inverse can be used for general control systems represented using Kronecker products. We begin by proving that the Kronecker product of two diagonal matrices is also diagonal. Given diagonal matrices A and B

$$A = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m_A,m_A} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & b_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{m_B,m_B} \end{bmatrix}$$

the Kronecker product

$$C = A \otimes B = \begin{bmatrix} a_{1,1}B & 0 & \cdots & 0 \\ 0 & a_{2,2}B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m_A,m_A}B \end{bmatrix}$$

can be seen to also be diagonal because every nonzero element of A is at position $[i, i]$ ($1 \le i \le m$), every nonzero element of B is at position $[j, j]$ ($1 \le j \le p$), and every nonzero element of C is at position $[i(p-1)+j, i(p-1)+j]$. Or more simply, every diagonal block of C is a diagonal matrix and therefore C must be a diagonal matrix.

With these basic properties in mind, we have the prerequisites to prove the UC inverse property. First consider the base case,

$$A_1^{-U} \otimes A_2^{-U} = (A_1 \otimes A_2)^{-U} \tag{20}$$

Now take a decomposition of matrix A, where for notational clarity we now use D and E instead of D_A and E_A:

$$A = D \cdot S \cdot E \tag{21}$$

and the UC generalized inverse of A is defined as:

$$A^{-U} = E^{-1} \cdot S^{-P} \cdot D^{-1} \tag{22}$$

Applying this to the left hand side of equation 20 gives

$$
\begin{aligned}
& A_1^{-U} \otimes A_2^{-U} \\
=\ & (E_1^{-1} \cdot S_1^{-P} \cdot D_1^{-1}) \otimes (E_2^{-1} \cdot S_2^{-P} \cdot D_2^{-1}) \quad (23)\\
=\ & (E_1^{-1} \otimes E_2^{-1})(S_1^{-P} \otimes S_2^{-P})(D_1^{-1} \otimes D_2^{-1}) \quad (24)\\
=\ & (E_1 \otimes E_2)^{-1}(S_1^{-P} \otimes S_2^{-P})(D_1 \otimes D_2)^{-1} \quad (25)
\end{aligned}
$$

Using the fact that $S_1^{-P} \otimes S_2^{-P} = (S_1 \otimes S_2)^{-P}$ gives

$$A_1^{-U} \otimes A_2^{-U} = (E_1 \otimes E_2)^{-1}(S_1 \otimes S_2)^{-P}(D_1 \otimes D_2)^{-1} \quad (26)$$

and the right-hand side of equation 20 is

$$
\begin{aligned}
(A_1 \otimes A_2)^{-U} &= [(D_1 \cdot S_1 \cdot E_1) \otimes (D_2 \cdot S_2 \cdot E_2)]^{-U} \quad (27)\\
&= [(D_1 \otimes D_2)(S_1 \otimes S_2)(E_1 \otimes E_2)]^{-U} \quad (28)
\end{aligned}
$$

where $D_1 \otimes D_2$ and $E_1 \otimes E_2$ are diagonal matrices. In order to apply equation 22, we need to prove the rows and columns of $S_1 \otimes S_2$ satisfy that the product is ± 1. Let S_1 and S_2 be represented as

$$S_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n_1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1,1} & a_{m_1,2} & \cdots & a_{m_1,n_1} \end{bmatrix}$$

$$S_2 = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n_2} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m_2,1} & b_{m_2,2} & \cdots & b_{m_2,n_2} \end{bmatrix}$$

where for any $1 \leq i_1 \leq m_1$, $1 \leq j_1 \leq n_1$, $1 \leq i_2 \leq m_2$, $1 \leq j_2 \leq m_2$, the matrix elements of S_1 and S_2 have the following property:

$$\prod a_{i_1,k} = \pm 1 \qquad (a_{i_1,k} \neq 0) \quad (29)$$

$$\prod a_{k,j_1} = \pm 1 \qquad (a_{k,j_1} \neq 0) \quad (30)$$

$$\prod b_{i_2,k} = \pm 1 \qquad (b_{i_2,k} \neq 0) \quad (31)$$

$$\prod b_{k,j_2} = \pm 1 \qquad (b_{k,j_2} \neq 0) \quad (32)$$

For every row $i_1(m_2 - 1) + i_2$ of

$$S_1 \otimes S_2 = \begin{bmatrix} a_{1,1}S_2 & a_{1,2}S_2 & \cdots & a_{1,n_1}S_2 \\ a_{2,1}S_2 & a_{2,2}S_2 & \cdots & a_{2,n_1}S_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1,1}S_2 & a_{m_1,2}S_2 & \cdots & a_{m_1,n_1}S_2 \end{bmatrix}$$

the product of its nonzero elements is (when $a_{i_1,k_1} \neq 0, b_{i_2,k_2} \neq 0$)

$$
\begin{aligned}
& \prod(a_{i_1,k_1}(\prod b_{i_2,k_2})) \quad (33)\\
=\ & \prod(a_{i_1,k_1}(\pm 1)) \quad (34)\\
=\ & \prod(a_{i_1,k_1}) \prod(\pm 1) \quad (35)\\
=\ & \prod(\pm 1) \quad (36)\\
=\ & \pm 1. \quad (37)
\end{aligned}
$$

The same holds analogously for every column of $S_1 \otimes S_2$, so equation 22 can be applied to equation 28 to obtain

$$
\begin{aligned}
& [(D_1 \otimes D_2)(S_1 \otimes S_2)(E_1 \otimes E_2)]^{-U} \\
=\ & (E_1 \otimes E_2)^{-1}(S_1 \otimes S_2)^{-P}(D_1 \otimes D_2)^{-1} \quad (38)
\end{aligned}
$$

and from equation 26 the following theorem can be concluded:

Theorem 1: $A_1^{-U} \otimes A_2^{-U} = (A_1 \otimes A_2)^{-U}$.

We now show that Theorem 1 can be used as the base case for a mathematical induction proof of the general case involving matrices $A_1 ... A_n$. Given

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^{-U} = A_1^{-U} \otimes A_2^{-U} \otimes \cdots \otimes A_n^{-U} \quad (39)$$

it is required to show

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_{n+1})^{-U} = A_1^{-U} \otimes A_2^{-U} \otimes \cdots \otimes A_{n+1}^{-U} \quad (40)$$

To simplify the equation, let $B = A_1 \otimes A_2 \otimes \cdots \otimes A_n$. Then

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_{n+1})^{-U} = (B \otimes A_{n+1})^{-U} \quad (41)$$

$$A_1^{-U} \otimes A_2^{-U} \otimes \cdots \otimes A_{n+1}^{-U} = B^{-U} \otimes A_{n+1}^{-U}. \quad (42)$$

Applying Theorem 1 while Letting $A_1 = B$ and $A_2 = A_{n+1}$ we obtain

$$(B \otimes A_{n+1})^{-U} = B^{-U} \otimes A_{n+1}^{-U} \quad (43)$$

from which we can expand to obtain the desired final result:

Theorem 2:
$(A_1 \otimes A_2 \otimes \cdots \otimes A_{n+1})^{-U} = A_1^{-U} \otimes A_2^{-U} \otimes \cdots \otimes A_{n+1}^{-U}$

Theorems 1 and 2 complete the set of UC inverse properties necessary to allow it to be used in place of the MP inverse in mechanical and robotic systems whenever unit consistency is required.

4 Discussion

It has previously been demonstrated that the UC inverse does in fact permit unit consistency to be rigorously preserved in a practical system, thus corroborating theoretical predictions. In many practical systems, however, there is a mix of consistency conditions that must be satisfied for different subsets of state variables. Most commonly, some variables demand unit consistency, i.e., system behavior should be invariant with respect to the choice of units on those variables, while other variables demand rotational consistency, i.e., system behavior should be invariant with respect to rotations of the Cartesian coordinate frame in which they are jointly defined.

In this paper, we demonstrated in a practical example that invariant system behavior can be guaranteed in accordance with theory in the case when different subsets of state variables must satisfy different consistency conditions. It is important to emphasize that although we examined a specific case involving a mix of unit-consistent and rotation-consistent state variables, the general approach is agnostic to the particular consistency conditions that are enforced, i.e., we could have chosen two different conditions or included different conditions for additional subsets of state variables and the same invariant behavior should be expected.

We also provided a formal proof that the unit-consistency properties of the UC inverse are preserved under applications of the Kronecker product. With this we believe the theory of consistent inverses is now largely complete.

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