New Fuzzy Aggregations. Part II: Associated Probabilities in the Aggregations of the POWA operator

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Abstract: - The Ordered Weighted Averaging (OWA) operator was introduced by R.R. Yager [58] to provide a method for aggregating inputs that lie between the max and min operators. In this article several variants of the generalizations of the fuzzy-probabilistic OWA operator - POWA (introduced by J.M. Merigo [27,28]) are presented in the environment of fuzzy uncertainty, where different monotone measures (fuzzy measure) are used as an uncertainty measure. The considered monotone measures are: possibility measure, Sugeno $\lambda-$additive measure, monotone measure associated with Belief Structure and capacity of order two. New aggregation operators are introduced: AsPOWA and SA-AsPOWA. Some properties of new aggregation operators are proved. Concrete faces of new operators are presented with respect to different monotone measures and mean operators. Concrete operators are induced by the Monotone Expectation (Choquet integral) or Fuzzy Expected Value (Sugeno integral) and the Associated Probability Class (APC) of a monotone measure. For the new operators the information measures – *Orness, Entropy, Divergence* and *Balance* are introduced as some extensions of the definitions presented in [28].

Key-Words: - mean aggregation operators, fuzzy aggregations, fuzzy measure, capacity of order, associated probabilites, most typical value, Finite Sugeno Averaging, Finite Choquet Averaging, body of evidence, possibility measure, fuzzy numbers, fuzzy decision making.

1 Introduction

Our research is concerned with quantitative-information analysis of the complex uncertainty and its use for modeling of more precise decisions with minimal decision risks from the point of view of systems approach. The main objects of our attention are 1) the analysis of Information Structures of experts knowledge, their uncertainty measure and imprecision variable, which was constructed in the Part I of this work; 2) the construction of instruments of aggregation operators, which condense both characteristics of incomplete information - an uncertainty measure and an imprecision variable in the scalar ranking values of possible alternatives in the decision making system. Some aspects of this problem are considered in current Part of our research.

In Section 2 some preliminary concepts are presented. Probability representations – Associated Probability Class (APC) of a monotone measure [5,37,39,42,44] is considered for different classes of a monotone measure. Concepts of the Most Typical Value (MTV) [18,19,41,42] of a compatibility function (membership function) of some imprecise variable with respect to some monotone measure is presented. The Fuzzy Expected value (FEV) [9] and Monotone Expectation (ME) [5] are interpreted as important MTVs of a compatibility function. The probability representations of ME and FEV are presented by the APC of a monotone measure. Also in this Subsection the associated probabilities representations are considered for the Choquet capacity of order two [7], possibility measure [11], Sugeno $\lambda-$additive measure [45] and a monotone measure associated with Dempster-Shafer Belief Structure [45].

In Section 3 new generalizations of the POWA operator (definition 4, Part I) are presented with respect to different monotone measures (insert of the probability measure) and different mean operators. New versions of the POWA operator are defined. AsPOWA operator is induced by the ME and SA-AsPOWA operator is induced by the FEV. In Subsection 3.3 the generalized variants of information measures – Orness, Entropy, Divergence and Balance are introduced for the new aggregation operators. Some properties of new operators are proved.

2 Associated probabilities of a monotone measure

When trying to functionally describe insufficient expert data, in many real situations the property of additivity remains unrevealed for a measurable representation of a set and this creates an additional restriction. Hence, to study such data, it is frequently better to use monotone measures instead of additive ones.

We introduce the definition of a monotone measure (fuzzy measure) [45] adapted to the case of a finite referential.

DEFINITION 1: *Let* $S=\{s_1,s_2,...,s_m\}$ *be a finite set and* g *be a set function* $g:2^S \Rightarrow [0,1]$. *We say* g *is a monotone measure on* S *if it satisfies*

(i) $g(\varnothing)=0;\ g(S)=1;$

(ii) $\forall A,B\subseteq S,\ if\ A\subseteq B,\ then\ g(A)\le g(B).$

A monotone measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity. Non-additive but monotone measures were first used in the fuzzy analysis in the 1980s [45] and is well investigated ([8, 15, 21-23, 37-39, 44, 45, 54-56, 62] and others).

A fuzzy integral is a functional which assigns some number or a compatibility value to each fuzzy subset when the monotone measure is taken as an uncertainty measure. As known ([10, 15, 18, 19, 25, 26, 37, 38, 45, 63] and others), the concept of a fuzzy integral condenses the information provided by a compatibility (or membership) function of a fuzzy set and a monotone measure. Having the monotone measure determined, we can estimate a fuzzy subset by the most typical compatibility value - most typical value (MTV) ([18, 19 ,41-45] and others) or a fuzzy average. As already known, fuzzy averages (MTVs) differ both in form and content from probabilistic–statistical averages and other numerical characteristics such as mode and median and others. Nevertheless, in some cases 'non-fuzzy' (objective) and 'fuzzy' (subjective) averages coincide ([18, 19, 41-45] and others). For a given set of fuzzy subsets with compatibility function values from the interval [0; 1], the fuzzy average determines the most typical representative compatibility value. From the point of our future presentations in the role of MTV we consider only two fuzzy statistics (integrals):

1. Monotone Expectation – ME (or Choquet Integral) and

2. Fuzzy Expected Value – FEV (or Sugeno Integral). So, we consider some aspects of a monotone measure in fuzzy statistics.

DEFINITION 2: *Assume* $S=\{s_1,s_2,...,s_m\}$ *is a set on which we have a monotone measure* g *and a function* $a:S\Rightarrow R_0^+$ *such that* $a(s_i)\equiv a_i\ge 0, i=1,2,...,m$.

Then

a) The aggregation

$$ME_g(a_1,a_2,...,a_m)\equiv$$
$$\equiv FCA(a_1,a_2,...,a_m)=\sum_{j=1}^{m} w_j a_{i(j)}\ , \tag{1}$$

Where $w_j=g(\{s_{i(1)},.....,s_{i(j)}\})-g(\{s_{i(1)},.....,s_{i(j-1)}\})$, $g(\{s_{i(0)}\})\equiv 0$, *is called a Finite Choquet Averaging (FCA) or Monotone Expectation (ME) operator. In the proceeding* $i(\cdot)$ *is index function such that* $a_{i(j)}$ *is the jth largest of the* $\{a_i\}_{i=1}^m$.

b) *The aggregation*

$$FEV_g(a_1,...,a_m)\equiv FSA(a_1,...,a_m)=$$
$$=a_{\max}\max_{j=1,m}\min\{a_{i(j)}/a_{\max}\ ;\ \hat{w}_j\}, \tag{2}$$

where $\hat{w}_j=g(\{s_{i(1)},s_{i(2)},...,s_{i(j)}\})$, $a_{\max}=\max_{i=1,m}\{a_i\}$ *is called a Finite Sugeno Averaging (FSA) or a Fuzzy Expected Value (FEV) operator.*

The ME always exists and is finite for each monotone measure g and some compatibility variable a. It is obvious that $ME_g(a)$ is a generalization of the mathematical expectation $E_P(a)$ and the ME of a non-negative function a with respect to a monotone measure g coincides with the mathematical expectation of a with respect to a probability measure that depends only on g and the ordering of the values of a.

Following the definition 2a the maximum number of probability distributions in ME (formula 1) coincides with the number of possible orderings or permutations in a set with m elements, that is, $m!$. Thus, it makes sense to associate the $m!$ probabilities to each monotone measure, provided that they are deduced from this monotone measure through the different possible orderings.

In general, the possible orderings of the elements of S are given by the permutations of a set with m elements, which form the group S_m.

DEFINITION 3 [5]: *The probability functions* P_σ *defined by*

$$P_\sigma\left(s_{\sigma(1)}\right)=g\left(\left\{s_{\sigma(1)}\right\}\right),....,$$
$$P_\sigma\left(s_{\sigma(i)}\right)=g\left(\left\{s_{\sigma(1)},....,s_{\sigma(i)}\right\}\right)-g\left(\left\{s_{\sigma(1)},....,s_{\sigma(i-1)}\right\}\right),....,$$
$$P_\sigma\left(s_{\sigma(m)}\right)=1-g\left(\left\{s_{\sigma(1)},....,s_{\sigma(m-1)}\right\}\right), \tag{3}$$

for each $\sigma=(\sigma(1),\sigma(2),...,\sigma(m))\in S_m$, *are called the associated probabilities and the Associated Probability Class (APC)* -$\{P_\sigma\}_{\sigma\in S}$ *of the monotone measure* g.

An interesting case is when the monotone measure is a probability. It is easy to prove that in this case, all associated probabilities are equal.

PROPOSITION 1 [5]: *A monotone measure* g *is a probability measure* $(g=p)$ *if and only if its m! associated probabilities coincide.*

The concept of duality of monotone measures is very important, since it permits one to obtain alternative representations of a piece of information. Monotone measures g_* *and* g^* are dual if $g_*(A)=1-g^*(\overline{A}),\forall A\subset S$. So, we will consider a monotone measure and its dual measure to contain the same information, but codified in a different way. The most remarkable case where different monotone measures provide the same *m*! probabilities, but ordered in a different way, is the case of dual monotone measures. Before exposing it in the following proposition, we need a definition:

DEFINITION 4: *We will say that two permutations* $\sigma,\sigma^*\in S_m$ *are dual if* $\sigma^*(i)=\sigma(m-i+1),\ i=1,...,m$.

PROPOSITION 2 [5]: *A necessary and sufficient condition for two monotone measures* g_* *and* g^* *to be dual is to have the same m! associated probabilities corresponding to dual permutations, that is,* $P_{*\sigma}=P^*_{\sigma^*}$, *if* σ *and* σ^* *are dual, where* P_* *and* P^* *are associated probabilities for the measures* g_* *and* g^* *respectively.*

An especially interesting class of monotone measures is the capacities of order two [7], because they cover a great number of monotone measures.

DEFINITION 5: *Let* (g_*,g^*) *be a pair of dual monotone measures:*

g_* *is a lower capacity of order two if and only if*
$$\forall\, A,B\subseteq S,\quad g_*(A\cup B)+g_*(A\cap B)\geq g_*(A)+.g_*(B);$$
g^* *is an upper capacity of order two if and only if*
$$\forall\, A,B\subseteq S,\quad g^*(A\cup B)+g^*(A\cap B)\leq g^*(A)+.g^*(B).$$

The most used classes of monotone measures such as belief and plausibility measures [35], necessity and possibility ones [11], λ-measures [45] and probabilities are capacities of order two.

PROPOSITION 3 [5]: *Let* (g_*,g^*) *be a pair of dual monotone measures. Then* g_* *is a lower capacity of order two (*g_* *is an upper capacity of order two, respectively) if and only if*

$$g_*(A)=\min_{\sigma\in S_m}P_\sigma(A)\ \ \forall\, A\subseteq X,$$
$$(\ g^*(A)=\max_{\sigma\in S_m}P_\sigma(A)\ \ \forall\, A\subseteq X,\ ). \tag{4}$$

So the main characteristic of a capacity of order two is that it only depends on the probabilities associated to such a measure, but does not depend on the permutations that generate them: we can regenerate the initial monotone measure by only knowing its associated probabilities, without the necessity to know the corresponding permutations. This characteristic makes the use of capacities of order two by means of associated probabilities especially easy.

Starting from this property, the following result is evident and valid for every monotone measure:

PROPOSITION 4 [5]: *If* $P_\sigma,\sigma\in S_m$, *are the associated probabilities to a monotone measure* g, *then for every* $a:X\to R_0^+$, *it holds*
$$\min_{\sigma\in S_m}E_{P_\sigma}(a)\leq ME_g(a)\leq\max_{\sigma\in S_m}E_{P_\sigma}(a). \tag{5}$$

PROPOSITION 5 [39]: *A necessary and sufficient condition for a pair of dual fuzzy measures* (g_*,g^*) *to be lower and upper capacities of order two, respectively, is that* $\forall\, a:X\to R_0^+$,
$$ME_{g_*}(a)=\min_{\sigma\in S_m}E_{P_\sigma}(a),\ ME_{g^*}(a)=\max_{\sigma\in S_m}E_{P_\sigma}(a). \tag{6}$$

Let $S_m^{(a)}$ $\left(S_m^{(a)}\subset S_m\right)$ be the subgroup of all permutations such that $\forall\,\sigma\in S_m^{(a)}$,
$$a\left(s_{\sigma(1)}\right)\geq a\left(s_{\sigma(2)}\right)\geq...\geq a\left(s_{\sigma(m)}\right). \tag{7}$$

Following Proposition 2 and Definitions 2-4 there exist some connections of mathematical expectations with respect to dual associated probability $P_{*\sigma};P^*{}_\sigma\left(\sigma\in S_m^{(a)}\right)$:

$$ME_{g_*}(a)=E_{P_{*\sigma}}(a)=\sum_{i=1}^{m}P_{*\sigma}\left(s_{\sigma(i)}\right)a\left(s_{\sigma(i)}\right),$$
$$ME_{g^*}(a)=E_{P^*_\sigma}(a)=\sum_{i=1}^{m}P^*_\sigma\left(s_{\sigma(i)}\right)a\left(s_{\sigma(i)}\right)=, \tag{8}$$
$$=\sum_{j=1}^{m}P_{*\sigma_*}\left(s_{\sigma_*(m-i+1)}\right)a\left(s_{\sigma_*(m-i+1)}\right)=E_{P_{*\sigma_*}}(a)$$

where $P_{*\sigma}$ and P_{σ}^{*} are associated probabilities for g_* and g^* monotone measures, respectively; σ and σ_* are dual permutations and a is symmetric.

2.1. Probability representation of the FEV

It clearly follows that (definition 2b) the FEV somehow 'averages' the values of the compatibility function a not in the sense of a statistical average but by cutting subsets of the α level, whose values of monotone measure g are either sufficiently 'high' or sufficiently 'low'. The FEV gives a concrete value of the compatibility function a, this value being the most typical characteristic of all possible values with respect to the monotone measure g, obtained by cutting off the 'upper' and 'lower' strips on the graph of $g(H_\alpha) = g(\{s / a(s) \ge \alpha\})$. Thus, the incomplete information carried by an imprecision variable a and an uncertain measure g is condensed in the FEV, which is the MTV of all compatibility levels of a. Following definition 2b for all permutation such that $\sigma \in S_m^{(a)}$ the FEV can be written by the associated probabilities of a lower capacity of order two g_* as

$$FEV_{g_*}(a) = a_{\max} \min_{\underline{j=1,m}} \min_{\sigma' \in S_m} \max\{a(s_{\sigma(i)} / a_{\max}); P_{*\sigma'}(A_i^{(\sigma)})\} \tag{9}$$

where $A_i^{(\sigma)} = \{s_{\sigma(1)}, s_{\sigma(2)}, ..., s_{\sigma(i)}\}$, $i = 1,..,m$.

Let (g_*, g^*) be a pair of a dual lower and upper capacities of order two. Using propositions 2, 3 and formula (9) the FEV can be written, $\forall \sigma \in S_m^{(a)}$:

$$FEV_{g_*}(a) = a_{\max} \min_{\underline{j=1,m}} \min_{\sigma' \in S_m} \max\{a(s_{\sigma(i)} / a_{\max}); P_{*\sigma'}(A_i^{(\sigma)})\},$$

$$FEV_{g^*}(a) = a_{\max} \max_{\underline{i=1,m}} \max_{\sigma' \in S_m} \min\{a(s_{\sigma(i)}) / a_{\max}; P_{\sigma'}^{*}(A_i^{(\sigma)})\} =$$
$$= a_{\max} \max_{\underline{i=1,m}} \max_{\sigma' \in S_m} \min\{a(s_{\sigma(i)}) / a_{\max}; P_{*\sigma'^*}(A_i^{(\sigma)})\}. \tag{10}$$

2.2. Dempster–Shafer Belief Structure and Its Associated Probabilities

The Theory of Evidence (Dempster–Shafer Belief Structure) ([11, 15, 22, 23, 25, 32, 37, 43, 56,59,62] and others) is a powerful tool which enables one to build:

1. Models of decisions and their risks' measures;
2. Aggregation operators in an uncertain environment and so on.

The Theory of Evidence is based on two dual monotone measures: Belief measures and Plausibility measures. These classes of monotone measures are subclasses of classes of dual lower and upper capacities of order two. This is easily provable after introduction of Belief and Plausibility measures ([22, 23] and others). Belief and Plausibility measures can be characterized by the set function:

$$\mathrm{m}: 2^S \Rightarrow [0;1], \tag{11}$$

which is required to satisfy two conditions:

(a) $\mathrm{m}(\varnothing) = 0$,

(b) $\sum_{B \in 2^S} \mathrm{m}(B) = 1$.

This function is called a Basic Probability Assignment (BPA). For each set $B \in 2^S$, the value $m(B)$ expresses the proportion that all available and relevant evidence supporting the claim that a particular element of S, whose characterization in terms of relevant attributes is deficient, belongs to the set B. This value $m(B)$, pertains solely to one set $-B$; it does not imply any additional claims regarding subsets of B. If there is some additional evidence supporting the claim that the element belongs to a subset of B, say $B_1 \subseteq B$, it must be expressed by another value $m(B_1)$ [23].

Let m be a PBA on S. The plausibility measure Pl associated to m is given by

$$\mathrm{Pl}(A) = \sum_{B \subset S:\, A \cap B \ne \varnothing} \mathrm{m}(B), \ \forall A \in 2^S$$

and the Belief measure Bel associated to m is given by

$$\mathrm{Bel}(A) = \sum_{B: B \subset A} \mathrm{m}(B), \ \forall A \in 2^S.$$

Inverse procedures are also possible. Given, for example, a Belief measure Bel, the corresponding BPA is determined for all $A \in 2^S$ by formula

$$\mathrm{m}(A) = \sum_{B: B \subseteq A} (-1)^{|A \setminus B|} \mathrm{Bel}(B), \tag{12}$$

where $|A \setminus B|$ is the cardinality of the set difference of A and B. If the Belief measure is also additive that is

$$\begin{gathered} Bel(A \cup B) = Bel(A) + Bel(B), \\ if \quad A \cap B = \varnothing, \quad A, B \in 2^S \end{gathered}, \tag{13}$$

then we obtain the classical probability measure [23].

Given a BPA, every set $A \in 2^S$ for which $m(B) > 0$ is called a focal element. The pair $\langle F_S, m \rangle$ where F_S denotes the set of all focal elements induced by m is called a Body of Evidence. Because Bel is a lower capacity of order two, then using proposition 3 and formulas (29) and (30) we receive probability representation of the BPA, $\forall A \in 2^S$, $\sigma \in S_m$:

$$\mathrm{P}_\sigma^{(Bel)}\left(s_{\sigma(i)}\right) = \sum_{\substack{B \in F_S : B \subseteq \{s_{\sigma(1)},..,s_{\sigma(i)}\} \\ B \cap \{s_{\sigma(i)}\} \neq \emptyset}} \mathrm{m(B)},$$

$$\mathrm{m(A)} \quad = \sum_{B \in F_S : B \subseteq A} (-1)^{|A \setminus B|} \min_{\sigma \in S_m} \mathrm{P}_\sigma^{(Bel)}(B),$$

(14)

where $\left\{P_\sigma^{(Bel)}\right\}_{\sigma \in S_m}$ are the associated probabilities of the monotone measure Bel.

2.3. Possibility Measure and Its Associated Probabilities

When the focal elements of a body of evidence $\langle F_S, m \rangle$ are required to be nested, $F = \left\{A_{j_1} \subset A_{j_2} \subset ... \subset A_{j_l}\right\}$, the associated belief and plausibility measures are called consonant [23]. The special branch of the evidence theory that deals only with bodies of evidence whose focal elements are nested is referred to as the possibility theory [11].

Special counterparts of Bel measures and Pl measures in the possibility theory are called necessity (Nec) measures and possibility (Pos) measures, respectively:

PROPOSITION 6 [23]: *Given a consonant body of evidence $\langle F_S, m \rangle$, the associated consonant belief (necessity) and plausibility (possibility) measures possess the following properties:*

$$Nec(A \cap B) = \min\{Nec(A); Nec(B)\} \ \ for \ \ all \ \ A, B \in 2^S,$$
$$Pos(A \cup B) = \max\{Pos(A); Pos(B)\} \ for \ \ all \ \ A, B \in 2^S. \tag{15}$$

PROPOSITION 7 [23]: *Every possibility measure Pos on 2^S can be uniquely determined by its possibility distribution function $\pi : S \Rightarrow [0,1]$; $\max_{s \in S} \pi(s) = 1$ via the formula:*

$$\forall A \in 2^S, \ Pos(A) = \max_{s \in A} \pi(s). \tag{16}$$

Assume the finite universe $S = \{s_1, s_2, ..., s_m\}$ is given and let $F_S = \{A_{j_1} \subset A_{j_2} \subset ... \subset A_{j_l}\}$ be some consonant body of evidence.

Let

$m_{j_i} \equiv m\left(A_{j_i}\right), \ i = 1,...,l;$

$\pi_i \equiv \pi(s_i), \ \pi_i \geq \pi_{i+1}; \ i = 1,...,m; \pi_1 = 1.$

Then, we have the l-tuple

$$m = \left\langle m_{j_1}, m_{j_2}, ..., m_{j_l} \right\rangle \tag{17}$$

and m-tuple

$$\pi = \left\langle \pi_1, \ \pi_2, ..., \pi_m \right\rangle. \tag{18}$$

It is easy to show that

$$\begin{cases} \pi_i = \sum_{\nu : s_i \in A_{j_\nu} \in F_S} m_{j_\nu}, \ \ i = 1,2,..,m \\ m_{j_i} = \pi_{j_i} - \pi_{j_{i+1}}, \ \ \pi_{j_{l+1}} \equiv 0, \ \ i = 1,2,...,l. \end{cases} \tag{19}$$

Let $\left\{P_\sigma^{(Pos)}\right\}_{\sigma \in S_m}$ be the associated probabilities class of a possibility measure Pos. Then, we have the following connection between $\{\pi_i\}$, $\left\{m_{j_i}\right\}$ and $\{P_\sigma\}_{\sigma \in S_m} : \forall \sigma \in S_m$

$$P_\sigma^{(Pos)}\left(s_{\sigma(i)}\right) = Pos\left(\left\{s_{\sigma(1)}, ..., s_{\sigma(i)}\right\}\right) - Pos\left(\left\{s_{\sigma(1)}, ..., s_{\sigma(i-1)}\right\}\right) =$$
$$= \max_{\nu = 1, i} \pi(s_{\sigma(\nu)}) - \max_{\nu = 1, i-1} \pi(s_{\sigma(\nu)}) =$$
$$= \max_{\nu = 1, i} \sum_{q : s_{\sigma(\nu)} \in A_{j_q} \in F_S} m_{j_q} - \max_{\nu = 1, i-1} \sum_{q : s_{\sigma(\nu)} \in A_{j_q} \in F_S} m_{j_q} =$$
$$= \begin{cases} 0, & otherwise \\ \sum_{q : s_{\sigma(i)} \in A_{j_q} \in F_S} m_{j_q} - \sum_{q : s_{\sigma(i')} \in A_{j_q} \in F_S} m_{j_q}, & if \ \sigma(i') < \sigma(i) \end{cases} \tag{20}$$

Since Pos is a capacity of order two, using proposition 5 we receive:

$$\pi_i = Pos(\{s_i\}) = \max_{\sigma \in S_m} \mathrm{P}_\sigma^{(Pos)}(\{s\}_i), \ \ i = 1,2,..,m, \tag{21}$$

$$m_{j_i} = \pi_{j_i} - \pi_{j_{i+1}} =$$
$$= \max_{\sigma \in S_m} \mathrm{P}_\sigma^{(Pos)}\left(\{s_{j_i}\}\right) - \max_{\sigma \in S_m} \mathrm{P}_\sigma^{(Pos)}\left(\{s_{j_{i+1}}\}\right), \ i = 1,2,..,l. \tag{22}$$

2.4. Monotone Measures Associated with a Belief Structure and Its Associated Probabilities

Let m be a BPA with a body of evidence $F_S = \{A_1, A_2, ..., A_q\}$. For each focal element A_j, $j = 1,..,q$, let W_j^0 be a weighting vector of dimension $|A_j|$ whose components $w_j^0(i)$ $\left(W_j^0 \equiv \left\langle w_j^0(1), ..., w_j^0(|A_j|)\right\rangle\right)$ satisfy the conditions $w_j^0(i) \in [0,1]$, $\sum_{i=1}^{|A_j|} w_j^0(i) = 1$. We shall call these the allocation vectors. In [56], it was shown that a set function $g : 2^S \to [0,1]$ defined by

$$g(A) = \sum_{j=1}^{q} \left[m(A_j) \cdot \sum_{i=1}^{|A_j \cap A|} w_j^0(i) \right], ... \forall A \in 2^S \tag{23}$$

is a monotone measure associated with the belief structure. Thus, by selecting a collection $W^0 = \left\{W_1^0, W_2^0, ..., W_q^0\right\}$ of allocation vectors, we can define a unique monotone measure associated with a belief structure. For example: if all the W_j^0 are such that $w_j^0(1)=1$, then the resulting monotone measure is the plausibility measure Pl. If all W_j^0 are selected such that $w_j^0(|A_j|)=1$, then this results in the belief measure Bel.

We have the following important proposition concerning all associated monotone measures with a belief structure.

PROPOSITION 8 [56]: *If g is any monotone measure generated from a collection of allocation vectors, then*

(a) $Bel(A) \le g(A) \le Pl(A)$ $\forall A \in 2^S$;

(b) *The Shapley Entropy of generated monotone measures coincide* $E_{Shapley}(Bel)=E_{Shapley}(g)=E_{Shapley}(Pl)$.

I.e. generated monotone measures have the same information but codified in a different way.

Now, we shall compute the associated probabilities of a monotone measure g associated with the belief structure: $\forall \sigma \in S_m, \forall i=1,2,...,m$.

$$P_\sigma(s_{\sigma(i)})=g(\{s_{\sigma(1)},...,s_{\sigma(i)}\})-g(\{s_{\sigma(1)},...,s_{\sigma(i-1)}\})=$$
$$=\sum_{j=1}^{q} m(A_j)\left[\sum_{\nu=1}^{|A_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|} w_j^0(\nu)-\sum_{\nu=1}^{|A_j\cap\{s_{\sigma(1)},...,s_{\sigma(i-1)}\}|} w_j^0(\nu)\right]= \quad (24)$$
$$=\sum_{A_j\in F_S:A_j\cap\{s_{\sigma(i)}\}\ne 0} m(A_j)w_j^0(|A_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|).$$

2.5. Sugeno λ-additive Monotone Measure and Its Associated Probabilities

DEFINITION 6 [45]: *A monotone measure $g_\lambda : 2^S \Rightarrow [0,1]$ $(\lambda > -1)$ is called a λ-additive monotone measure if for any $A,B \in 2^S$, $A \cap B = \varnothing$,*
$$g_\lambda(A\cup B)=g_\lambda(A)+g_\lambda(B)+\lambda g_\lambda(A)\cdot g_\lambda(B).$$
(25)

It is easy to verify that for any $A \in 2^S$

$$g_\lambda(A)=\frac{1}{\lambda}\left\{\prod_{s_i\in A}(1+\lambda g_i)-1\right\}, \quad (26)$$

where $0<g_i\equiv g(\{s_i\}),\ i=1,...,m;\ \lambda>-1$ is the parameter with following normalization condition:

$$\frac{1}{\lambda}\left\{\prod_{s_i\in S}(1+\lambda g_i)-1\right\}=1. \quad (27)$$

Note, that $g_0(\lambda=0)$ is a probability measure if $\sum_{s_i\in S} g_i=1$.

It is easy to prove that the λ-additive monotone measure g_λ is a capacity of order two and $g_\lambda^* = g_{-\lambda/(1+\lambda)}$.

Due to (26), (27) and (3), we can write the class of associated probabilities for the λ-additive monotone measure g_λ for any $\sigma \in S_m$ as

$$\mathrm{P}_\sigma(s_{\sigma(i)})=g_\lambda(\{s_{\sigma(i)}\})\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\})), \quad (28)$$

or, more exactly, as

$$\mathrm{P}_\sigma(s_i)=g_\lambda(\{s_i\})\prod_{j=1}^{i(\sigma)-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\})), \quad (29)$$

where $i=1,2,...,m,\ \sigma\in S_m;\ i(\sigma)$ is the location of s_i in the permutation σ (if $i(\sigma)=1$, then $\prod_{j=1}^{0}\equiv 1$).

3. Associated Probabilities' Aggregations in the POWA Operator

Different approaches were developed by the authors, which constructed aggregation operators with respect to a monotone measure, where I1-I6 and other levels of Information Structure (definition 7, Part I) were considered ([1-4, 6, 9, 10, 13, 14, 16, 17, 20, 21, 24-34, 36-44, 46-55, 57-61, 63] and others). But for the POWA or FPOWA-type operators (definitions 4 and 5, Part I) Information Structures on the levels I5 and I6 (or weighted OWA operators constructed on the basis of a monotone measure) were not investigated. So, we leave the Information Structures I1-I4 and go to the levels of I5 and I6. In this paper we consider the level I5 and we will consider the level I6 in the Part III of this work.

It is important that in the aggregation operators POWA and FPOWA the both nature of incomplete information: 1. An uncertain measure (probability distribution $\{p_i\}$) and 2. An imprecision variable (random variable (a) or fuzzy variable ($\tilde{a}$)) are condensed in the outcome values, which gives us more credibility to use these aggregation operators in applications.

In this Section we define new generalization of the POWA operator where more general measure of uncertainty – monotone measure (fuzzy measure) is used instead of probability measure in the role of uncertainty measure.

3.1. AsPOWA operators induced by the ME

Let on the states of nature $S=\{s_1,s_2,...,s_m\}$ be given some monotone measure $g:2^S\Rightarrow[0,1]$ instead of probability measure $P=\{p_1,p_2,...,p_m\}$, $p_i=\mathrm{P}(s_i)$. There exist many aggregations in the decision making systems when we use monotone measure g as a measure of fuzzy uncertainty ([10, 15, 18, 19, 24-26, 36, 37, 39, 40-43] and others) the definition of which was given in Section 2. In Section 2 the FEV and ME were defined along with their probability representations by associated probability class (APC) $\{\mathrm{P}_\sigma\}_{\sigma\in S_m}$, where the number of probability distributions on S is equal to $k=m!$. We have k values of mathematical expectations for random or fuzzy-random variable a - $\{\mathrm{E}_{\mathrm{P}_\sigma}(a)\}_{\sigma\in S_m}$, where

$$\mathrm{E}_{\mathrm{P}_\sigma}(a)=\sum_{i=1}^m a_i P_\sigma(s_i),\ \sigma\in S_m. \quad (30)$$

So, we will focus on the use of $m!$ mathematical expectations in the POWA operator, instead of one expectation $\mathrm{E}_{\mathrm{P}}(a)=\sum a_i p_i$, as a more usual extension of this operator.

Let $M:R^k\Rightarrow R^1,\ k=m!$ be some deterministic mean aggregation function with symmetricity, boundedness, monotonicity and idempodency properties (see the definition in the Section 2, Part I). Let $a:S\Rightarrow R_0^+$ be some variable.

DEFINITION 7: *An associated POWA operator - AsPOWA of dimension m is a mapping $AsPOWA:R^m\Rightarrow R^1$, that has an associated objective weighted vector W of dimension m such that $w_j\in[0,1]$ and $\sum_{i=1}^m w_j=1$, some uncertainty measure - monotone measure $g:2^S\Rightarrow[0,1]$ with associated probability class $\{\mathrm{P}_\sigma\}_{\sigma\in S_m}$, and is defined according the following formula:*

$$AsPOWA(a_1,a_2,...,a_m)=\beta\sum_{j=1}^m w_j b_j +$$

$$+(1-\beta)\cdot M\left(\sum_{i=1}^m a_i \mathrm{P}_\sigma(s_i)\Big/\sigma\in S_m\right)=$$

$$=\beta\sum_{j=1}^m w_j b_j+(1-\beta)\cdot M\left(E_{P_{\sigma_1}}(a),\ E_{P_{\sigma_2}}(a),...,\ E_{P_{\sigma_k}}(a)\right), \quad (31)$$

where b_j is the jth largest of the $\{a_i\},i=1,...,m.$

It is easy to prove that in general cases of operator M the AsPOWA operator is induced by the ME:

PROPOSITION 9: *Let M be the Min operator, then AsPOWA operator may be written as:*

$$AsPOWA\min(a_1,a_2,...,a_m)=$$

$$=\beta\sum_{j=1}^m w_j b_j+(1-\beta)\cdot\min_{\sigma\in S_m}\left(\sum_{i=1}^m a_i\mathrm{P}_\sigma(s_i)\Big/\sigma\in S_m\right), \quad (32)$$

and if monotone measure g is a lower capacity of order two, then in the AsPOWAmin operator the second addend coincides with ME_g :

$$AsPOWA\min(a_1,a_2,...,a_m)=$$

$$=\beta\cdot OWA(a_1,a_2,...,a_m)+(1-\beta)\cdot ME_g(a_1,a_2,...,a_m) \quad (33)$$

PROPOSITION 10: *Let M be the Max operator, then AsPOWA operator may be written as:*

$$AsPOWA\max(a_1,a_2,...,a_m)=$$

$$=\beta\sum_{j=1}^m w_j b_j+(1-\beta)\cdot\max_{\sigma\in S_m}\left(\sum_{i=1}^m a_i\mathrm{P}_\sigma(s_i)\right), \quad (34)$$

and if monotone measure g is an upper capacity of order two, then in the AsPOWAmax operator the second addend coincides with ME_g :

$$AsPOWA\max(a_1,a_2,...,a_m)=$$

$$=\beta\cdot OWA(a_1,a_2,...,a_m)+(1-\beta)\cdot ME_g(a_1,a_2,...,a_m). \quad (35)$$

These proofs are easy if we use the results of proposition 5 (formula (6)).

PROPOSITION 11: *Let M be any mean aggregation operator and in AsPOWA operator monotone measure g is a probability measure. Then AsPOWA and POWA operators coincide.*

$$AsPOWA(a_1,a_2,...,a_m)=POWA(a_1,a_2,...,a_m). \quad (36)$$

Proof: As known the associated probabilities of probability measure coincide (see proposition 1). Using the property of idempotency of operator M $\left(M\left(E_{P_1},E_{P_2},...,E_{P_m}\right)\equiv E_P\right)$,

because $p_i\equiv p,\ i=1,...,k;\ E_{p_i}=E_p$ and $M(E_p,E_p,...,E_p)=E_p$, then AsPOWA removes to the POWA (formula (9), Part I).

PROPOSITION 12: *If g_* and g^* are dual monotone measures on 2^S, then AsPOWA operators constructed on basis g_* and g^* coincide:*

Proof: Using symmetricity of operator M and results of proposition 2 it is easy to prove this proposition: consider AsPOWA operator for the lower monotone measure g_*

$$AsPOWA_*(a_1,a_2,...,a_m)=$$
$$=\beta\sum_{j=1}^{m} w_j b_j+(1-\beta)M\left(E_{P_{*\sigma_1}}(a),E_{P_{*\sigma_2}}(a),...,E_{P_{*\sigma_k}}(a)\right)=$$
$$=\beta\sum_{j=1}^{m} w_j b_j+(1-\beta)M\left(E_{P^*_{\sigma_1}}(a),E_{P^*_{\sigma_2}}(a),...,E_{P^*_{\sigma_k}}(a)\right)=$$
$$=AsPOWA^*(a_1,a_2,...,a_m),$$

where $\{P_{*\sigma_i}\}_{i=1}^{k}$ is the associated probability class for the measure g_* and $\{P^*_{\sigma_i}\}_{i=1}^{k}$ is the associated probability class for the measure g^*.

Now we consider different variants of the AsPOWA operator induced by the ME with respect to different classes of monotone measures. Following the Section 2 associated probabilities' formulas were presented for different classes of monotone measures. For example: a) possibility measure (Subsection 2.3); b) monotone measure associated with a belief structure (Subsection 2.4); c) Sugeno λ-additive monotone measure (Subsection 2.5). Therefore there exist many combinatorial possibilities for the analytical construction of concrete faces of the AsPOWA operator for concrete classes of a monotone measure and concrete operator M induced by the ME. But this procedure is omitted here. We will consider some of them:

1) Consider AsPOWAmax for the Sugeno λ-additive monotone measure g_λ. Using formulas (34) and (28), we receive:

$$AsPOWA\max(a_1,a_2,...,a_m)=$$
$$=\beta\cdot\sum_{j=1}^{m} b_j w_j+(1-\beta)\cdot$$
$$\cdot\max_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda(\{s_{\sigma(i)}\})\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot a_{\sigma(i)}\right\} \tag{37}$$

2) Analogously we may construct the face of AsPOWAmin:

$$AsPOWA\min(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m} b_j w_j+$$
$$+(1-\beta)\cdot\min_{\sigma\in S_m}\left\{\begin{array}{l}\sum_{i=1}^{m}\left[g_\lambda(\{s_{\sigma(i)}\})\cdot\prod_{j=1}^{i-1}(1+\lambda g_\lambda(\{s_{\sigma(j)}\}))\right]\cdot\\ \cdot a_{\sigma(i)}\end{array}\right\} \tag{38}$$

3) Following Subsection 2.4 we consider the AsPOWAmin and AsPOWAmax operators for the monotone measure associated with the belief structure. Using formulas (32),(33) and (24) we construct new variants of the AsPOWA operator:

$$AsPOWA\max(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m} b_j w_j+(1-\beta)\cdot$$
$$\cdot\max_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[\sum_{F_j\in F_S:F_j\cap\{s_{\sigma(i)}\}\neq\varnothing}\frac{m(F_j)w_j^0}{\left(|F_j\cap\{s_{\sigma(1)},...,s_{\sigma(i)}\}|\right)}\right]\cdot a_{\sigma(i)}\right\} \tag{39}$$

$$AsPOWA\min(a_1,a_2,...,a_m)=\beta\cdot\sum_{j=1}^{m} b_j w_j+(1-\beta)\cdot$$
$$\cdot\min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[\sum_{F_j\in\Im:F_j\cap\{s_{\sigma(i)}\}\neq\varnothing} m(F_j)w_j^0\left(|F_j\cap\{S_{\sigma(1)},...,S_{\sigma(i)}\}|\right)\right]\cdot a_{\sigma(i)}\right\} \tag{40}$$

3.2. AsPOWA operators induced by the FEV

In this Subsection we define new generalizations of the POWA operator induced by the Sugeno Averaging Operator - Fuzzy Expected Value (FEV) with respect to probability measure - P. Analogously definition 7 (formula (31)) but difference is that Mathematical Expectation operator $E_p(.)$ is changed by the $FEV_P(.)$.

DEFINITION 8: *A Sugeno Averaging POWA operator SA-POWA of dimension m is a mapping $SA-POWA: R^m \Rightarrow R_0^+$ that has an associated weighting vector W of dimension m such that $w_j\in[0,1]$ and $\sum_{j=1}^{m} w_j=1$ according to the following formula:*

$$SA-POWA(a_1,a_2,...,a_m)=$$
$$=\beta\cdot\sum_{j=1}^{m} b_j w_j+(1-\beta)\cdot FEV_P(a_1,a_2,...,a_m)=$$
$$\overset{\Delta}{=}\beta\cdot\sum_{j=1}^{m} w_j a_{i(j)}+(1-\beta)\cdot\max_{l=1,m}\{a_l\}\max_{j=1,m}\left\{\min\left[a'_{i(j)},w_j^P\right]\right\}, \tag{41}$$

where $b_j=a_{i(j)}$ is the j-th largest of the $\{a_i=a(s_i)\geq 0\}$, $i=1,2,...,m$; on S there exists probability distribution $\{p_i=P(s_i)\}$ with $\sum_{j=1}^{m} p_i=1,\quad 0\leq p_i\leq 1$;

$$w_j^P = P\left(\left\{s_{i(1)}, s_{i(2)}, ..., s_{i(j)}\right\}\right) = \sum_{l=1}^{j} p_{i(l)} \text{ and}$$

$$a'_{i(j)} = \frac{a_{i(j)}}{\max\limits_{l=1,m}\{a_l\}}.$$

On the basis of the definitions 2b and 8 analogously to the definition 7 we may generalize the POWA operator induced by the FEV with respect to some monotone measure g .

DEFINITION 9: *A Sugeno Averaging AsPOWA operator SA-AsPOWA of dimension m is mapping $SA-AsPOWA: R^m \Rightarrow R_0^+$, that has an associated objective weighted vector W of dimension m such that $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$; some monotone measure $g: 2^S \to [0,1]$ with associated probability class $\{P_\sigma\}_{\sigma \in S_m}$, according the following formula:*

,

$$SA-AsPOWA(a_1, a_2, ..., a_m) =$$
$$= \beta \cdot \sum_{j=1}^{m} w_j a_{i(j)} + (1-\beta) \cdot M \begin{pmatrix} FEV_{P_{\sigma_1}}(a_1, a_2, ..., a_m), \\ FEV_{P_{\sigma_2}}(a_1, a_2, ..., a_m), ..., \\ FEV_{P_{\sigma_k}}(a_1, a_2, ..., a_m) \end{pmatrix} \quad (42)$$

where

$$FEV_{P_\sigma}(a_1, a_2, ..., a_m) = \max_{l=1,m}\{a_l\} \max_{j=1,m} \min\{a'_{i(j)}; w_j^{P_\sigma}\}, \quad (43)$$

and $w_j^{P_\sigma} = P_\sigma\left(\left\{s_{i(1)}, ..., s_{i(j)}\right\}\right) = \sum_{i=1}^{m} P_\sigma(\{s_{i(j)}\})$,

$$a'_{i(j)} = \frac{a_{i(j)}}{\max\limits_{l}\{a_l\}}, \ \forall \sigma \in S_m.$$

Now we consider SA-AsPOWA operators induced by the FEV with respect to $M = Max$ and $M = Min$ averaging operators:

$$SA-AsPOWA\max(a_1, a_2, ..., a_m) =$$
$$= \beta \cdot \sum_{j=1}^{m} w_j a_{i(j)} +$$
$$+ (1-\beta) \cdot \max_{l=1,m}\{a_l\} \max_{\sigma \in S_m}\left[\max_{j=1,m}\left\{\min\left[a'_{i(j)}, w_j^{P_\sigma}\right]\right\}\right] \quad (44)$$

$$SA-AsPOWA\min(a_1, a_2, ..., a_m) =$$
$$= \beta \cdot \sum_{j=1}^{m} w_j a_{i(j)} +$$
$$+ (1-\beta) \cdot \max_{l=1,m}\{a_l\} \min_{\sigma \in S_m}\left[\min_{j=1,m}\left\{\max\left[a'_{i(j)}, w_j^{P_\sigma}\right]\right\}\right] \quad (45)$$

It is easy to prove the propositions analogously to propositions 9-12. But these propositions are omitted here.

3.3. Information Measures of the AsPOWA and SA-AsPOWA Operators

Analogously to [28] (see Section 3, Part I) now we extend the definitions of the information measures for the AsPOWA and SA-AsPOWA operators:

DEFINITION 10: *The Orness measure of the AsPOWA operator is the extension of the formula (13),* Part I:

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot M\left[\sum_{j=1}^{m} P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right) \Big/ \sigma \in S_m\right] \quad (46)$$

For AsPOWAmax we receive:

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot \max_{\sigma \in S_m}\left[\sum_{j=1}^{m} P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\right] \quad (47)$$

but for AsPOWAmin we have:

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot \min_{\sigma \in S_m}\left[\sum_{j=1}^{m} P_{\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\right] \quad (48)$$

Constructing the *Orness* measure of the SA-AsPOWA operator induced by the FEV we receive the analogous extension.

DEFINITION 11: *The Orness measure of the SA-AsPOWA operator is the extension of the formula (13)*,Part I:

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot M\left[\max_{j=1,m} \min\left\{\frac{m-\sigma(j)}{m-1}; w_j^{P_\sigma}\right\} \Big/ \sigma \in S_m\right] \quad (49)$$

For example, for the AsPOWAmax operator we have:

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot \max_{\sigma \in S_m}\left[\max_{j=1,m} \min\left\{\frac{m-\sigma(j)}{m-1}, w_j^{P_\sigma}\right\}\right] \quad (50)$$

and for AsPOWAmin :

$$\alpha\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = \beta \cdot \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right) + (1-\beta) \cdot \min_{\sigma \in S_m}\left[\min_{j=\overline{1,m}} \max\left\{\frac{m-\sigma(j)}{m-1}, w_j^{P_\sigma}\right\}\right] \quad (51)$$

DEFINITION 12: *The entropy (the dispersion) H of the AsPOWA operator of the amount of information is defined as:*

$$H\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = -\left\{\begin{array}{l}\beta \cdot \sum_{j=1}^{m} w_j \ln(w_j) + \\ +(1-\beta) \cdot M\left[\sum_{j=1}^{m} P_{\sigma(j)} \ln(P_{\sigma(j)}) \middle/ \sigma \in S_m\right]\end{array}\right\} \quad (52)$$

For example, if we have AsPOWAmax operator, then

$$H\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = -\left\{\begin{array}{l}\beta \cdot \sum_{j=1}^{m} w_j \ln(w_j) + \\ +(1-\beta) \cdot \max_{\sigma \in S_m}\left[\sum_{j=1}^{m} P_{\sigma(j)} \ln(P_{\sigma(j)}) \middle/ \sigma \in S_m\right]\end{array}\right\} \quad (53)$$

and for AsPOWAmin:

$$H\left(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m\right) = -\left\{\begin{array}{l}\beta \cdot \sum_{j=1}^{m} w_j \ln(w_j) + \\ +(1-\beta) \cdot \min_{\sigma \in S_m}\left[\sum_{j=1}^{m} P_{\sigma(j)} \ln(P_{\sigma(j)}) \middle/ \sigma \in S_m\right]\end{array}\right\} \quad (54)$$

DEFINITION 13: *The divergence measure Div has the following face:*

$$Div\left(\hat{P}_1, \hat{P}_2, ..., \hat{P}_m\right) = \beta\left\{\sum_{j=1}^{m}\left(\frac{m-j}{m-1} - \alpha(W)\right)^2\right\} + (1-\beta)\left\{M\left[\sum_{j=1}^{m} P_{\sigma(j)} \cdot \left(\frac{m-\sigma(j)}{m-1} - \alpha(P_\sigma)\right)^2 / \sigma \in S_m\right]\right\} \quad (55)$$

where $\alpha(W)$ *is an Orness measure of the OWA operator*

$$\alpha(W) = \sum_{j=1}^{m} w_j \left(\frac{m-j}{m-1}\right)$$

and $\alpha(P)$ *is an Orness measure of associated probabilities' aggregations:*

$$\alpha(P_\sigma) = \sum_{j=1}^{m} P_{\sigma(j)} \left(\frac{m-\sigma(j)}{m-1}\right). \quad (56)$$

Analogously to definition 13 we may construct the concrete analytical forms of the measure Div for AsPOWAmax and AsOWAmin and other operators with respect to different monotone measures (Here these formulas are omitted).

DEFINITION 14: *The Balance parameter of the AsPOWA operator has the following extension*

$$Bal\left(\hat{P}_1, \hat{P}_2, ..., \hat{P}_m\right) = \beta \sum_{j=1}^{m} w_j \left(\frac{m+1-2j}{m-1}\right) + (1-\beta) M\left[\sum_{j=1}^{m} P_{\sigma(j)} \left(\frac{m+1-2\sigma(j)}{m-1}\right) \middle/ \sigma \in S_m\right] \quad (57)$$

The Bal of the AsPOWAmax and AsPOWAmin operators and the H, Div, Bal parameters of the SA-AsPOWA operator may be written analogously definitions 10-14, but are omitted here.

4. Conclusions

New generalizations of the POWA operator were presented with respect to monotone measure's associated probability class (APC) and induced by the Choquet or Sugeno integrals (finite cases). There exist many combinatorial variants to construct faces or expressions of generalized operators: AsPOWA, and SA-AsPOWA for concrete mean operators (Mean, Max, Min and so on) and concrete monotone measures (Choquet capacity of order two, monotone measures associated with belief structure, possibility measure and Sugeno λ – additive measure). Some properties of new operators and their information measures (*Orness*, *Enropy*, *Divergence and Balance*) are proved. But only some variants (AsPOWAmax, AsPOWAmin and others) are presented, the list of which may be longer that it is presented in the paper. So, other presentations of new operators and properties of information measures will be considered in our future research. The new generalizations of the FPOWA operator in the fuzzy environment with respect to monotone measures will be considered in the Part III of this work, where a practical example will be constructed for the illustration of the properties of generalized operators.

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