On the Gravitational Instability of a Compressible Gas Layer

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Abstract: - The stability of the equilibrium of a compressible gas placed in a limited region in a gravity field is investigated. The same temperature is maintained on all boundaries of the region. The stability of the static equilibrium of a compressible gas is analyzed in a linear approximation. The obtained data are supplemented by the results of solving a system of complete nonlinear equations describing the flows of a compressible gas. The characteristics of the obtained nonstationary solution are discussed.

Key-Words: - Gas, turbulence, compressibility, linear analysis, gravitational force, numerical simulation

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1 Introduction

It is known [1] that in an incompressible medium considered within the framework of the Boussinesq approximation, in the absence of a vertical temperature gradient, only monotonically decaying motions are observed. Moreover, with this approach, the density depends only on the temperature, and the dependence on pressure is neglected [1]. Thus, in the absence of a vertical temperature gradient, there is also no heterogeneity in density in the vertical direction, which is the reason for the attenuation of the perturbations of the static solution.

But in a compressible medium, the dependence of density on pressure becomes significant [2]. Pressure plays an active role here and its change, in principle, can generate non-uniformity in density in the vertical direction with the development of instability of the static solution. However, due to the extremely poor study of slow flows of a compressible medium (small Mach numbers) and technical difficulties, the question of the feasibility of such a scenario is not discussed in the literature in any way [3].

However, the calculations performed in this work convincingly showed that when instability of the static regime develops in a compressible medium under the action of gravity, some heating of the medium is observed, which makes the flow in such a medium physically similar to Rayleigh-Benard convection, which occurs when a layer of gas or liquid is heated from below [4].

However, in the overwhelming majority of works, Rayleigh-Benard convection is considered as an incompressible fluid flow within the Boussinesq approximation [1,4]. And convection in a compressible gas medium, even in such a relatively simple case, is poorly understood. It has been shown that on a laboratory scale (with a layer height of the order of several centimeters), the compressibility of the medium is weakly manifested and convection of a compressible gas medium can be considered within the Boussinesq approximation as an incompressible fluid flow. However, when the height of the region exceeds the critical value (17.3 cm for air under normal conditions), a relatively large pressure variation causes a significant change in density, which greatly changes the characteristics of convective processes [3,5]. For example, the possibility of developing adiabatic processes makes it possible for convective instability to develop even with a stable density stratification of the medium [2].

The features of convection in a compressible medium have been discussed in a number of monographs and discussions [6-10]. It is traditionally considered that gas compressibility under convection at laboratory scales is insignificant, and it is essential only at large (planetary) scales. In this case, both scales (laboratory and planetary) are considered as asymptotic and their intersection is not taken into account.

In [8], the planetary atmosphere is treated as a compressible medium in which the flow is assumed to be adiabatic. It is shown that convective flow in the atmosphere is stable in the absence of a vertical temperature gradient.

In [11], gas convection in a horizontal layer with horizontal boundaries free of tangential stresses is considered analytically in the linear approximation and numerically in the nonlinear approximation. It is argued that the static solution is stable in the absence of a vertical temperature gradient and becomes unstable with sufficiently strong heating from below.

The results of the present work convincingly show that at a sufficiently large height of the layer of compressible gas located in the gravity field, instability of the static regime develops. However, the amplitudes of the disturbances of the static solution are very small and this circumstance explains why these flows have not been studied earlier.

To illustrate the importance of studying such flows, we point out their significance for the issue of explosion safety when storing hydrocarbons in large tanks, for example, at automobile filling stations.

An explosive situation occurs when the tank is almost empty, but some small amount of hydrocarbon remains at the bottom of the tank. In the presence of any flow, the fuel (vapors of vaporized hydrocarbon) mixes with the oxidizer (air), forming a potentially explosive gas-vapor medium. The key point here is the question of the presence or absence of movement (mixing) of the medium, and its intensity does not play a special role in this context. The formation of an explosive mixture has been studied in many studies [12-15].

In this paper, the stability of the equilibrium of a compressible gas placed in a limited region in a gravity field is investigated. All boundaries of the region are assumed to be rigid with a no-slip condition for velocity and isothermal. First, the stability of the static equilibrium of a layer of compressible gas is analyzed in the linear approximation. The obtained data are supplemented by the results of solving a system of complete nonlinear equations describing the flows of compressible gas. The characteristics of the emerging nonstationary regime are discussed.

2 Numerical Model and Problem Statement

The convective flow of a compressible, viscous and heat-conducting gas in a gravity field can be described by the following system of equations [1,16]:

$$\rho_t + \rho\, div\vec{u} + u\cdot\rho_x + v\cdot\rho_y = M\,\nabla^2(\rho-\rho_h),$$
$$u_t + u\cdot u_x + v\cdot u_y = -\frac{1}{\gamma\rho}(\rho T)_x + M\,(\frac{4}{3}u_{xx} + u_{yy} + \frac{1}{3}v_{xy}),$$
$$v_t + u\cdot v_x + v\cdot v_y = -\frac{1}{\gamma\rho}(\rho T)_y + M\,(v_{xx} + \frac{4}{3}v_{yy} + \frac{1}{3}u_{xy}) - C_F, \quad (1)$$
$$T_t + u\cdot T_x + v\cdot T_y = \frac{M}{\Pr}\nabla^2 T - \frac{\gamma-1}{\gamma}T\, div\vec{u},\ P = \rho T.$$

Here u, v, P, ρ and T are dimensionless components of velocity, pressure, density and temperature, $M = \nu/((\gamma T_0 R)^{0.5}H) = 4.608\cdot 10^{-8}\cdot H^{-1}$ is the Mach number, where the velocity calculated from kinematic viscosity is related to the adiabatic speed of sound, T_0 = 300˚K is taken as the characteristic value for temperature, the selected values of specific gas constant R = 287 J/(kg·K), adiabatic index γ = 1. 4, kinematic viscosity $\nu = 16\cdot 10^{-6}$ m^2/s and Prandtl number Pr = ν/χ = 0.71 correspond to air, where χ denotes the gas diffusivity and $C_F = gH/(\gamma RT_0) = 8.130\cdot 10^{-5}\cdot H$ is the hydrostatic compressibility and g is standard acceleration of free fall. As the length scale we chose the height of the region H, for temperature and density - their values T_0 and ρ_0 at the lower horizontal boundary, for the velocity - adiabatic speed of sound $(\gamma RT_0)^{0.5}$, for the pressure - $R\rho_0T_0$ and time - $H/(\gamma RT_0)^{0.5}$. The dependence of viscosity and thermal conductivity coefficients on temperature is neglected in the calculations. The height of the flow region in the calculations varies from 0.003 m to 0.5 m.

The equation for temperature (the fourth equation of system (1)) in the case of a region of low altitude asymptotically transforms into the equation for temperature in an incompressible medium in the Boussinesq approximation [1].

Figure 1 shows the formulation of the problem in dimensionless form. The horizontal size of the region, referred to the vertical, is equal to π in all simulations. All vertical and horizontal boundaries of the region are considered rigid with the no-slip condition for velocity and isothermal.

This problem has a static solution:

$$u = 0, v = 0, T = 1, \rho_h(y) = e^{-\gamma C_F y}, \rho_h(y) = 1 - \gamma C_F y.$$

The relationships for density are derived from the system of equations (1) taking into account the absence of motion, the smallness of the value of C_F, and the equality of the dimensionless value of density to 1 at the lower horizontal boundary.

The calculations were performed using the explicit scheme in time, and since the appearance of shock waves in the solution was not expected, a non-divergent formulation of the system of equations was used. The convective nonlinear and diffusion terms were approximated by the monotonic scheme of A.A. Samarskii [4], and thus the numerical method used was of the first order of approximation in time and the second order in space.

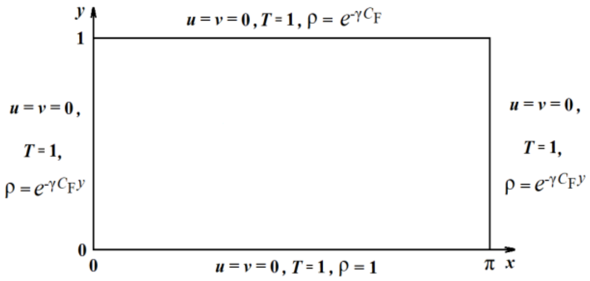

Fig. 1. Formulation of the problem.

All calculations were performed on a grid of (241·81) nodes with a dimensionless time step of 0.01. Test calculations on more detailed space and time grids showed sufficient accuracy and stability of the algorithm used.

All calculations were carried out near the stability threshold with the Reynolds number value as follows:

$$\mathrm{Re} = \sqrt{2Ek/\pi}\cdot M^{-1},$$

which did not exceed values of the order of 10^{-2}. Here Ek denotes the total kinetic energy of the entire moving mass of gas [2].

Due to the low velocity of flows, the main contribution to the pressure variation is made by its hydrostatic component [3].

Note, that the height of the region for clarity is always a dimensional value.

3 Linear Stability Analysis of a Static Solution

For infinitely small perturbations of the static solution from system (1) we can obtain (for simplicity of presentation we use the same notations) system (2), which is given below.

We will consider the solutions of system (2) in the form that is usually used in studying the stability of convective flows [1]:

$$(\rho,u,v,T) = (\rho_0,u_0,v_0,T_0)\,e^{-\lambda t}e^{i(\alpha x+\beta y)}.$$

$$\begin{aligned}
&\rho_t + div\vec{u} - \gamma C_F v = M\,\nabla^2\rho,\\
&u_t + (\rho_x + T_x)/\gamma = M\,(\tfrac{4}{3}u_{xx} + u_{yy} + \tfrac{1}{3}v_{xy}),\\
&v_t + (-\gamma\, C_F T + T_y + \rho_y)/\gamma = M\,(v_{xx} + \tfrac{4}{3}v_{yy} + \tfrac{1}{3}u_{xy}),\\
&T_t + \frac{\gamma-1}{\gamma}div\vec{u} = \frac{M}{\mathrm{Pr}}\nabla^2 T.
\end{aligned} \qquad (2)$$

When deriving system (2), the hydrostatic compressibility CF were considered small compared to 1.

Here λ, u_0, v_0, ρ_0 and T_0 are complex constants, α and β are real constants and the amplitudes of the disturbances increase for real part $\lambda_r < 0$ and decay for $\lambda_r > 0$. In this section, the solution is considered to be periodic in the horizontal and vertical directions with wave numbers α and β. This formulation of the problem is physically close to the formulation of the problem of flow in a region with free horizontal boundaries [11].

From system (2) we can obtain a system of equations for the amplitudes:

$$\begin{aligned}
&-\lambda\rho_0 + i(\alpha u_0 + \beta v_0) - \gamma C_F v_0 + M\,(\alpha^2+\beta^2)\rho_0 = 0,\\
&-\lambda u_0 + \frac{i\alpha}{\gamma}(\rho_0 + T_0) + M(\tfrac{4}{3}\alpha^2 u_0 + \beta^2 u_0 + \tfrac{1}{3}\alpha\beta\, v_0) = 0,\\
&-\lambda v_0 - C_F T_0 + i\,\beta(T_0+\rho_0)/\gamma + M(\alpha^2 v_0 + \tfrac{4}{3}\beta^2 v_0 + \tfrac{1}{3}\alpha\beta u_0) = 0,\\
&-\lambda T_0 + (\gamma-1)\,i\,(\alpha u_0 + \beta v_0)/\gamma + M\,(\alpha^2+\beta^2)T_0/\mathrm{Pr} = 0.
\end{aligned} \qquad (3)$$

3.1 Development of disturbances in the absence of gravity

Let us consider the solutions of the systems of equations (2) and (3) in a special case, in the absence of gravity $C_F = 0$. Then, instead of (3), we obtain a simpler system of equations:

$$\begin{aligned}
&-\lambda\rho_0 + i(\alpha u_0 + \beta v_0) + M\,(\alpha^2+\beta^2)\rho_0 = 0,\\
&-\lambda u_0 + \frac{i\alpha}{\gamma}(\rho_0 + T_0) + M(\tfrac{4}{3}\alpha^2 u_0 + \beta^2 u_0 + \tfrac{1}{3}\alpha\beta\, v_0) = 0,\\
&-\lambda v_0 + i\,\beta(T_0+\rho_0)/\gamma + M(\alpha^2 v_0 + \tfrac{4}{3}\beta^2 v_0 + \tfrac{1}{3}\alpha\beta u_0) = 0,\\
&-\lambda T_0 + (\gamma-1)\,i\,(\alpha u_0 + \beta v_0)/\gamma + M\,(\alpha^2+\beta^2)T_0/\mathrm{Pr} = 0.
\end{aligned} \qquad (4)$$

System (4) was written in matrix form and from the condition that the determinant of the system is equal to zero, an algebraic equation of the fourth order was derived to determine λ:

$$\lambda^4 - \frac{13}{3}MS\lambda^3 + \frac{S(7M^2\gamma^2S + 2\gamma - 1)}{\gamma^2}\lambda^2 - \frac{MS^2(5M^2\gamma^2S + 4\gamma - 2)}{\gamma^2}\lambda + \frac{1}{3}M^2S^3\frac{4M^2\gamma^2S + 6\gamma - 3}{\gamma^2} = 0,\ S = \alpha^2+\beta^2.$$

Despite some cumbersomeness, the resulting equation has four solutions, which are written out analytically and divided into two groups:

$$l_{1,2} = M S, \quad l_{3,4} = \frac{7M}{6} S \pm \frac{i}{g} \sqrt{S(2g - 1)}.$$

It can be found that the first two roots correspond to a two-parameter family of solutions that are monotonically damped under the action of viscosity, where C_1 and C_2 are two arbitrary constants:

$$r_0 = C_1, \; T_0 = - C_1, \; u_0 = 0, \; v_0 = 0;$$
$$r_0 = 0, \; T_0 = 0, \; u_0 = C_2, \; v = - a\, C_2 / b.$$

This solution describes the flow of a viscous incompressible fluid, since the continuity equation $\alpha u+\beta v = 0$ ($u_x + v_y = 0$) is satisfied exactly here.

In this part, the situation is similar to that observed during convection of an incompressible fluid in the Boussinesq approximation. The obtained solutions determine the convective mode, since they are the ones that, in the presence of a vertical temperature gradient and gravity, lead to the development of convective motion in an incompressible fluid in the Boussinesq approximation. Note that in a compressible medium, the calculated neutral curve also corresponds to the convective mode [2,3,5].

However, the second group of solutions (roots $\lambda_{3,4}$) corresponds to motions of a more complex structure.

To avoid cumbersome calculations, we will limit ourselves to a numerical example, calculating the solution with the choice of specific values of the parameters $\alpha = 3$, $\beta = \pi$, $H = 0.5$ and $\gamma = 1.4$.

Carrying out obvious simplifications associated with discarding small terms, we obtain:

$$l = i \times 4.16284, \; T_0 = 0.4317, \; r_0 = 1.511, \; u_0 = 1, \; v_0 = 1.0472.$$

The solution written out defines a rapidly oscillating motion of a compressed gas, since the continuity equation $\alpha u + \beta v = 0$ ($u_x + v_y = 0$) is not satisfied here. This solution defines a thermoacoustic mode, since it corresponds to thermoacoustic waves, which are analogs of pressure waves.

As methodical considerations have shown, the oscillation frequency of the thermoacoustic mode is determined by the imaginary part of the roots λ3,4 and is determined only by the Poisson adiabatic index and the wave number $S^{0.5} = (\alpha^2 + \beta^2)^{0.5}$. Dependence on other parameters, such as the height of the region, the presence or absence of gravity or heating, is practically absent here.

It can be shown that the propagation speed of the thermoacoustic wave is equal to the adiabatic speed of sound. Or, in other words, the characteristic time scale of the thermoacoustic mode is equal to 1.

3.2 Linear Analysis with Gravity

Now let us consider the development of linear disturbances taking into account the gravity force $C_F > 0$. Similar to the procedure described above, a system of equations for the amplitudes is derived from system (3), the resulting system is rewritten in matrix form, and from the equality of the system determinant to zero, we obtain an equation for the increment λ. The resulting fourth-order algebraic equation for λ is solved numerically.

Test calculations have shown that the solutions corresponding to the convective mode in the absence of heating are always damped, however, what is very interesting and unusual, the solutions corresponding to the thermoacoustic mode can become increasing at a sufficiently large height of the region.

Fig. 2 shows the real part of the increment of the solution corresponding to the thermoacoustic mode λ_{3r} at $H = 0.5$ m and $\beta = \pi$ as a function of α. It is evident that in the range of wave numbers $0 \le \alpha < 8.6$ the solution corresponding to the thermoacoustic mode increases in time. The fastest growth of the solution is observed at $\alpha = 0$.

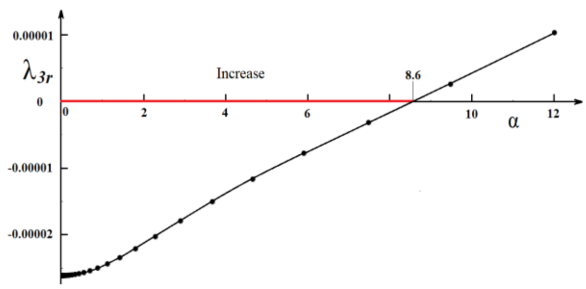

Fig. 2. Real part of the thermoacoustic mode growth rate as a function of α.

Figure 3 shows the real part of the solution growth rate corresponding to the thermoacoustic mode λ_{3r} at $\alpha = 0$ and $\beta = \pi$ as a function of H. It can be seen that the thermoacoustic mode becomes increasing at a region height greater than 0.1 m.

4 Results of Numerical Simulation

Now let us consider the results of numerical modeling using the complete nonlinear system of equations (1).

Fig. 4 shows the velocity field at $H = 0.5$ m, the two-vortex flow structure is clearly visible. Gas particles descend along the vertical boundaries, where the density is highest, and rise in the center of the region, where the density of the medium is lowest.

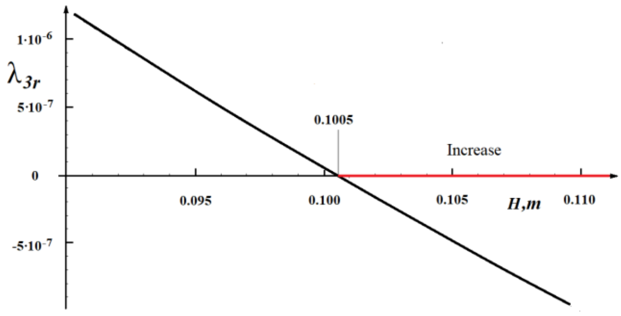

Fig. 3. Real part of the thermoacoustic mode growth rate as a function of H at $\alpha = 0$.

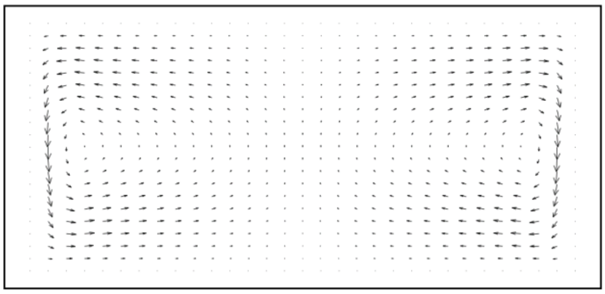

Fig. 4. Velocity field.

A small speed of movement (about 0.1 μm/s) corresponds to the Reynolds number shown in Fig. 5 as a function of the height of the region H. The signs in Fig. 5 show the results of numerical calculations.

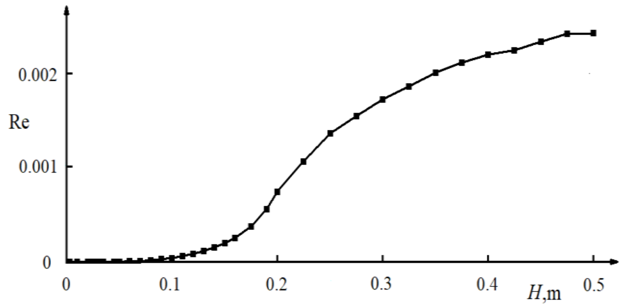

Fig. 5. Reynolds number.

Figure 6 shows the dimensionless temperature profile at H = 0.5 m, obtained by averaging the temperature field along the horizontal coordinate x. The shape of the temperature profile reflects the fact that heating is observed inside the region, with a maximum temperature of about 0.26 μK.

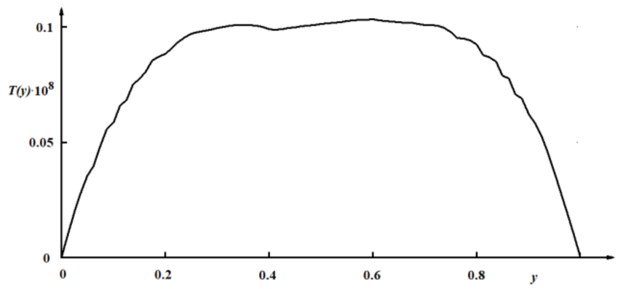

Fig. 6. Temperature profile.

We emphasize that inside the region the gas density decreases according to the relationship between the density and temperature deviations $\Delta\rho = -\Delta T$ [2], which follows from the equation of state. The density profile, taken with a minus sign, coincides with graphical accuracy with the temperature profile in Fig. 6.

In Fig. 7 the total kinetic energy Ek calculated over the entire region is shown as a function of time. The given dependence is large-scale, no periodicity is observed. The characteristic time scale of this large-scale motion is five orders of magnitude larger than the time scale of the thermoacoustic mode.

In Fig. 8 the energy spectrum of the dependence of Ek on time is shown. It is evident that the large-scale simplicity of the dependence of the kinetic energy on time is deceptive, since the given spectrum is complex and similar to turbulent. The power law -5/3 shown in Fig. 8 was observed with varying degrees of accuracy in all calculations at $H > 0.2$m.

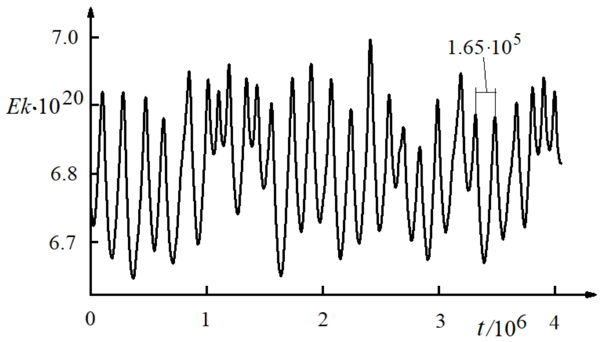

Fig. 7. Kinetic energy as a function of time.

5 Discussion

As linear analysis shows, the equilibrium state of a compressible gas layer in a gravity field is stable at a region height of less than 0.1 m. However, calculations performed using a complete nonlinear system of equations show that non-stationary regimes are also observed at a lower region height.

It is possible that this is due to the limited applicability of the performed linearization of the original system of equations at a low height of the region. But, the final answer to the question about the stability boundary of the static solution requires additional research.

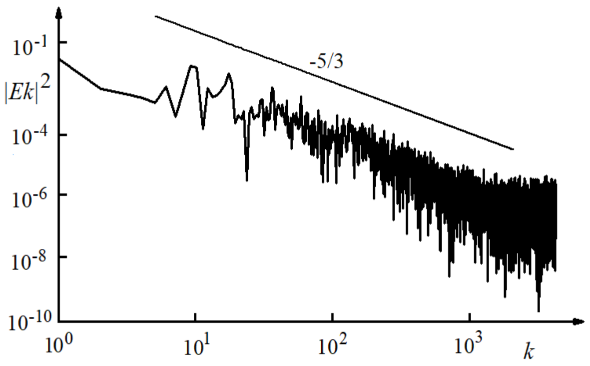

Fig. 8. Kinetic energy spectrum.

The linear stability analysis of the static solution shows that the rapidly oscillating thermoacoustic mode becomes unstable. However, the resulting flow is large-scale and its characteristic scale is at least five orders of magnitude larger than the characteristic time scale of the thermoacoustic mode. The time dependence of the kinetic energy is also large-scale, but this simplicity is deceptive. In fact, the kinetic energy spectrum is complex and resembles a turbulent one. At a sufficiently high altitude of the region, the energy spectrum of the time dependence of the kinetic energy corresponds to a power law of -5/3 with greater or lesser accuracy, the cause of which, as well as the energy processes of the formation of a large-scale flow from a small-scale (in time) require additional research.

In order to partially reduce the influence of small solution amplitudes on the accuracy of the obtained solution, it seems important in the long term to perform a similar series of calculations for the full nonlinear system of equations, but written in deviations from the static solution. It is possible that such an approach will be more effective in calculations with a small height of the region, where extremely small amplitudes of deviations from the static solution are observed.

6 Conclusions

In this paper, we study the stability of the equilibrium of a compressible gas placed in a limited region in a gravity field. All boundaries of the region are considered rigid and isothermal.

The constancy of temperature at all boundaries and the absence of unstable density stratification determine the absence of solutions corresponding to the convective mode. However, the results of the linear analysis show that when the height of the region is greater than 0.1 m, solutions corresponding to the rapidly oscillating thermoacoustic mode become increasing.

The development of instability of solutions corresponding to the thermoacoustic mode leads to the formation of a large-scale flow, with a characteristic time scale five orders of magnitude greater than the initial thermoacoustic one.

A study of the time dependence of the kinetic energy of the flow shows that the large-scale simplicity of the forming flow is deceptive. In fact, the spectrum of kinetic energy is complex and resembles turbulent.

The results obtained show that, contrary to the standard idea that the compressibility of a gaseous medium manifests itself only when moving in it at speeds of the order of sound or at a large height of the gas layer in a gravity field [7], the compressibility of a gaseous medium can play a significant role even when considering disturbances of an equilibrium static solution of infinitely small amplitude with zero gravity.

Let us present considerations explaining the development of instability of a compressible gas layer in a gravity field.

The greatest pressure in a gas layer is always observed at the lower horizontal boundary. The gas particles there are compressed and, accordingly, have the greatest potential energy.

The state of the system with the greatest potential energy is unstable, therefore the particle rises upward with a partial transition of potential energy into the kinetic energy of internal waves and into the internal energy of the gas with an increase in temperature.

It should be noted that there are technical devices that use physical principles close to those described in this paper [17]. We are talking about the perpetually running clock of J. Cox, which showed the time from 1774 to 1961 and was never wound manually. The role of the engine in it was performed by a mercury barometer, in which mercury, under the action of atmospheric pressure, rose from a glass container at the bottom of the clock up a glass tube. At the same time, the spring located inside the clock was compressed, storing energy for the operation of the clock. The driving force here is the daily pressure difference. Whereas in the present work, the driving force is the dependence of hydrostatic pressure on altitude, which, in the presence of compressibility of the medium, leads to the development of instability.

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