Study of the Barometric Formula for the Earth's Atmosphere

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*Abstract***:** A new barometric formula is proposed, applicable at large distances and free from a contradiction – the paradox with the acceleration of gravity vanishing at infinity. When compared with experimental data, an advantage was revealed for the case of a nonequilibrium equation of state corresponding to a Poisson ratio equal to 3.

Key-Words: - Earth's atmosphere, adiabatic temperature change, new barometric formula

Received: April 9, 2024. Revised: November 27, 2024. Accepted: December 27, 2024. Published: March 5, 2025.

1 Introduction

The barometric formula, or exponential atmosphere formula, has been known since the time of Laplace

$$n = n_0 \exp\left(-\frac{Mgh}{RT}\right).$$

It connects the concentration n of an isothermal ideal gas at a certain height h at a given temperature T with its concentration n_0 at zero height h=0, where M is the molar mass of gas particles, g is the acceleration of gravity (acceleration of the gravitational field), and R is the universal gas constant.

In practice, it is sufficient to know the dependence of concentration on potential energy in the force field. In this case, the barometric formula takes the form of the well-known Boltzmann distribution for a gas with an equation of state at a fixed temperature with pressure $p = \frac{mn}{M}RT$,where m is the mass of the molecules.

The derivation of the barometric formula is given in many physics textbooks (see, for example, [1, 2]). The barometric formula has generalizations for plasma processes [3-6] and for the state of the Fermi gas [6-10]. However, it is not applicable for large distances. This is indicated, for example, in textbooks [1, 2]. In this paper, we have attempted to solve this problem below. taking into account nonequilibrium processes, the consideration of which was developed in works [11-12], and to describe experimental data for a standard atmosphere [13]

2 Problem Formulation

The barometric formula describes the change in pressure and atmospheric concentration with increasing altitude above the Earth's surface at not very large distances from the surface. This is widely used in measuring altitude using special devices called altimeters, which are used in aviation and mountaineering.

However, with increasing distance from the surface, the acceleration of gravity decreases.

$$g = g_0 \frac{R_Z^2}{(R_Z + h)^2}\,, \tag{1}$$

where the acceleration of gravity near the Earth's surface is $g_0 = 9.81$ m/s2, R_Z-=6370 km is the radius of the Earth, h is the height above the Earth's surface. This leads to a paradox. According to the Boltzmann distribution and the barometric formula, the concentration of particles in the atmosphere (T is the temperature, R is the universal gas constant, M is the molar mass)

$$n = n_0 \exp\left(-\frac{Mgh}{RT} \right) \tag{2}$$

and should tend to n_0 at $h \to \infty$, since according to formula (1) the acceleration g tends to zero. This results in a contradiction. If the concentration is finite at infinity, then the atmosphere has infinite mass.

This means that formula (2) in its usual form is not applicable at large distances. This is written about in the courses of L.D. Landau and E.M. Lifshitz on theoretical physics [1] and the course of W. Greiner et al. [2]. It is indicated that at large distances there is no thermodynamic equilibrium and the molecules of the atmosphere simply fly apart.

3 Problem Solution

However, we can apply the Boltzmann distribution at sufficiently large distances if we take into account the change in temperature, its decrease, with distance. Considering the atmosphere as an adiabatic system, we can write the adiabatic equation [1,2]

$$TV^{\gamma-1} = T_0 V_0^{\gamma-1}, \tag{3}$$

where $V = \frac{4}{3}\pi\left(R_Z + h\right)^3 - \frac{4}{3}\pi R_Z^3 \approx \frac{4}{3}\pi\left(R_Z/3 + h\right)^3$ is the volume of the atmosphere at a distance h from the Earth's surface, T_0 and V_0 are the temperature and volume near the Earth's surface, γ is the adiabatic index.

3.1 New barometric formula

Next, using the usual reasoning for deriving the barometric formula, we have for the pressure difference p at the height dh

$$dp = -\rho g dh\,, \tag{4}$$

where, according to the Mendeleev-Clapeyron equation, the density is $\rho = \frac{pM}{RT}$. Substituting expressions (1) and (3) for acceleration and temperature into formula (4), we obtain after integration for pressure (p_0 is the pressure at sea level)

$$p = p_0 \exp\left(-\frac{Mg_0}{RT_0} R_Z \frac{\left((1+3h/R_Z\right)^{3\gamma\text{-}4} - 1)}{3\gamma - 4} \right). \tag{5}$$

and concentrations (n_0 is the concentration at sea level)

$$n = n_0 \exp\left(-\frac{Mg_0}{RT_0} R_Z \frac{\left((1+3h/R_Z\right)^{3\gamma\text{-}4} - 1)}{3\gamma - 4} \right). \tag{6}$$

Here in formulas (5) and (6) there is no longer a paradox at large distances h with the acceleration g vanishing according to formula (1). When h tends to infinity, the concentration n and pressure p vanish. The mass of the atmosphere is finite.

3.2 Comparison with experimental data

We compared the calculations with the experimental data [13]. Thus, Fig. 1 shows the change in the relative concentration of the atmosphere with distance in km. The experimental points [13] are marked with circles. The dashed-dotted curve 3 corresponds to the case of an isothermal atmosphere (formula (2)), which differs significantly at large distances from the experimental data and leads to a constant concentration at infinity. Dashed curve 2 corresponds to formula (6) and the adiabatic decrease in temperature according to the formula (3). with an adiabatic index of $\gamma = 5/3$. This corresponds to a monatomic gas. This agrees better with the experimental data. It is even better (solid curve 1) if the Poisson ratio - the adiabatic index is chosen to be equal to 3 (see [11-12]).

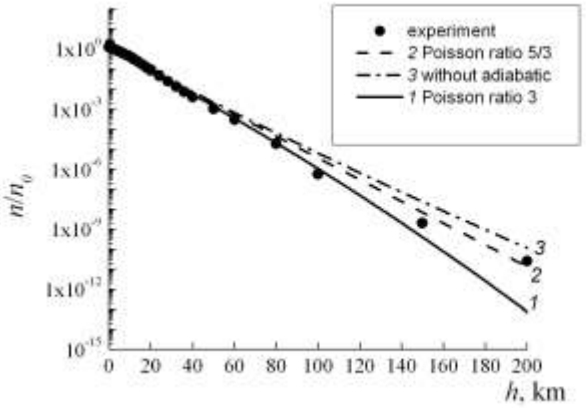

Fig.1.Change in the relative concentration of the Earth's atmosphere with altitude

It corresponds to non-equilibrium motion and equilibrium in the radial direction. There is no contradiction with the acceleration vanishing at infinity for these last two cases. The concentration and pressure vanish at infinity.

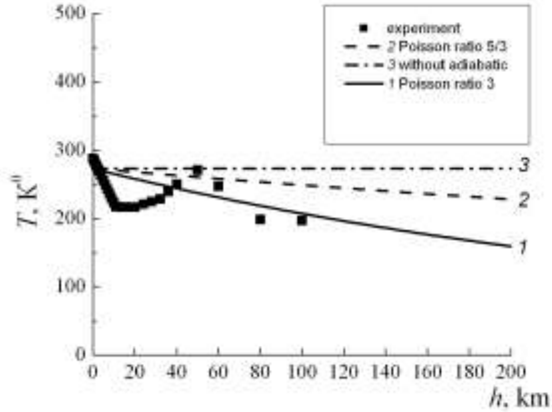

Fig.. 2. Change in the temperature of the Earth's atmosphere with altitude.

Fig. 2 shows the change in temperature with distance . The experimental points [13] are marked with squares. The dashed-dotted line 3 corresponds to the case of an isothermal atmosphere and does not coincide with the experimental data. The dashed curve 2 corresponds to the temperature decreasing adiabatically with the Poisson ratio equal to 5/3 according to formula (3). The solid curve 1 corresponds to the change in temperature according to formula (3) with the Poisson ratio equal to 3, i.e. the nonequilibrium case. Although our model does not take into account temperature fluctuations at different layers of the atmosphere, as can be seen from the experimental data, the decrease in temperature on average with height is correctly conveyed by formula (3), especially for curve 1 with an adiabatic index equal to 3.

4 Conclusion

Thus, we have proposed a new barometric formula (5)-(6), applicable at large distances and free from a contradiction – the paradox with the acceleration of gravity vanishing at infinity. When compared with experimental data, an advantage was revealed for the case of a nonequilibrium equation of state corresponding to a Poisson ratio equal to 3. Taking into account nonequilibrium processes is also important for very large distances, which we have not considered here. The arguments of the authors [1,2] will be important here.

References:

[1] L.D. Landau and E.M. Lifshitz, *Statistical Physics. Part I.* (Pergamon New York, 1980)
[2] W. Greiner, L. Neise, and Stoecker, *Thermodynamics and Statistical Mechanics*. (Springer 1995)
[3] C Sack and H Schamel, Evolution of a plasma expanding into vacuum. *Plasma Phys. Control. Fusion,* Vol. 27, 1985, p. 717; DOI 10.1088/0741-3335/27/7/002
[4] R. Z. Sagdeev, The 1976 Oppenheimer lectures: Critical problems in plasma astrophysics. Turbulence and nonlinear waves, *Rev. Mod. Phys.,* Vol. 51, 1979, pp.1-20; https://doi.org/10.1103/RevModPhys.51.1
[5] A.E. Dubinov, Mathematical tricks for pseudopotentials in the theories of nonlinear waves in plasma,. *Phys. Plasmas,* Vol. 29, 2022, p.020901; https://doi.org/10.1063/5.0078573
[6] A.E. Dubinov and I.D. Dubinova,. How can one solve exactly some problems in plasma theory, *J. Plasma Phys*. Vol. 71, No. 5, 2005, pp. 715–721; DOI: 10.1017/S0022377805003788
[7] J.Kinast, A.Turlapov, J.E.Thomas, Q.Chen, J. Stajic, and K. Levin, Heat capacity of a strongly interacting Fermi gas, *Science* Vol, 307, 2005, pp.1296-9; doi: 10.1126/science.1109220

[8] A.E. Dubinov and A. A. .Dubinova, Nonlinear theory of ion-acoustic waves in an ideal plasma with degenerate electrons, *Plasma Phys. Rep.,* Vol..33, 2007, pp..859–870; https://doi.org/10.1134/S1063780X0710007

[9] U. M. Abdelsalam, W.M. Moslem, and P. K.Shukla, Ion-acoustic solitary waves in a dense pair-ion plasma containing degenerate electrons and positrons. *Phys. Lett. A,* Vol. 372, *2008. pp. 4057–*4061;.https://doi.org/10.1016/j.physleta.2008.02.086

[10] A.E. Dubinov and M.A. Sazonkin, Nonlinear theory of ionic sound waves in a hot quantum-degenerate electron–positron–ion plasma, *J. Exper. Theor. Phys. Vol.* 111, 2010, pp. 865–876; □ DOI: https://doi.org/10.1134/S1063776110110178

[11] A.T.D’yachenko, Klein–Gordon Equations, Equations of Relativistic Quantum Hydrodynamics, and Quantum Shocks in Describing Collisions of Atomic Nuclei, *Physics of Atomic Nuclei,* , Vol. 87 2024, No. 5, pp. 646–658; DOI: 10.1134/S106377882470056X

[12] A. T. D’yachenko, K. A. Gridnev, and W. Greiner, Calculation of heavy ion collisions within the framework of the modified hydrodynamic approach, *J. Phys.G: Nucl. Part. Phys.*Vol. 40, 2013, p. 085101; doi:10.1088/0954-3899/40/8/085101

[13] *The atmosphere is standard.. The Great Soviet Encyclopedia:* In 30 Vol. (Set of 31 Books.) In Russian.(3rd edition of Moscow. Soviet Encyclopedia, 1978).