Stability and Stabilization of the Solidification Front for Melt Flow in Cylindrical Channel with Phase Change on a Wall. Part 1

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Abstract: - Theoretical research of the problems on boundary control of phase transition is conducted by creation of physical and mathematical models of the corresponding physical phenomena. The models are constructed, whenever possible, simple but reflecting main features of the studied phenomena. Complex mathematical models for the most general cases are realized on computer. So, in a task about stabilization of a garnissage the regions of system's instability are calculated on computer in a wide range of the varied parameters, and the values of coefficient of feedback control for the regulator intended for suppression of unstable modes of the boundaries' oscillations are defined. In a view of complexity of the considered magneto-hydrodynamic and thermal phenomena the most influential factors are considered, which have a principal importance in the studied systems. The constructed physical and mathematical models may be useful in further developments for the solution of more complex practical tasks. Under control of the boundary form of phase transition by means of systems of automatic control on the interface connected with the regulator the impedance boundary conditions are stated. And, as the regulator is, as a rule, connected to a considerable power source, the reverse influence of object on the regulator is insignificant and can be neglected that substantially simplifies a task. The results may be of interest for the problem of the wall protection with artificial garnissage.

Key-Words: - Control, Suppression, Instability, Stabilization, Solidification, Melt, Flow, Channel, Wall

1 Introduction

Problems of stability and stabilization (in case of instability) for the boundaries of phase transition are considered accounting many factors modeling various cases of real physical conditions. It is possible to refer multiple layers of walls of channels which may contain some layers of various materials of different thickness and physical properties, influence of convection on stability of boundaries of phase transition, dependence of the proceeding processes on physical properties of melts, casual and regular perturbations of a thermo-hydrodynamic condition of the medium in and out of the considered system, etc. to such factors [1-4].

In case of system's instability the device of automatic control of the heat fluxes [1,2] based on the phenomenon that perturbation of a boundary of phase transition leads to the perturbation of a magnetic field causing corresponding change of current in a winding is used. This current, after strengthening in the operating chain, directs secondary current in thin a skin-layer near the interfacial boundary separating the phases and the Joule heat fluxes cause suppression of the corresponding perturbations. For each harmonic mode of oscillation the impedance boundary condition of a form is obtained:

$$\frac{dT_{m,k}}{d\vec{n}} = G_{m,k} T_{m,k}, \qquad (1)$$

where $G_{m,k}$ is the coefficient of the feedback control system for suppression of the oscillation mode with the wave numbers m and k. The parameter $G_{m,k}$, as shown in [2], may have any real values. The boundary condition (1) can be considered as the law of feedback of the operating system. For control of several harmonicas in a control system on each mode there is the winding with individually adjustable coefficient of feedback that allows achieving high degree of resolution.

From all control methods for the boundaries of phase transition of electro-conductive liquids the greatest distribution in practical applications was gained by the electromagnetic method. Electromagnetic fields are used in metallurgy and electric welding in various purposes: for intensification the heat- and mass transfer processes (excitation of oscillations of the phase transition boundaries and management of melt circulation by means of alternating and running fields), for management of magneto-hydrodynamic and thermal instabilities in processes of the MHD-technologies, etc. Problems of stability and stabilization (in case of instability) for the boundaries of phase transition are considered accounting many factors modeling various cases of real and physical conditions.

2 Problem of instability of the phase transition boundaries

Increase of efficiency of metallurgical and electro-welding processes is often limited to overcoming need of different magneto-hydrodynamic instabilities, in particular, instability of the phase transition boundaries (fronts of crystallization) of the melts and diverse liquids. For example, instability of a thin layer (film) of a solid phase of the metal called garnissage and intended for protection of walls of the metallurgical units against destruction, doesn't allow using effectively artificial garnissage [2-4], known in metallurgy mostly as the negative phenomenon [5]. Instability of a garnissage layer worsens also quality of melt due to its pollution with material of the channel walls or crucibles of metallurgical units.

Magneto-hydrodynamic systems with existence of boundaries of phase transition (solidification) are systems with the distributed parameters, which behavior in space and time is described by the partial differential equations (PDE). From the theory of automatic control of processes in continua it is known that quite often unstable linear object can't be stabilized with just programmed influence, whereas control by the principle of feedback, at which the level of operating influence is connected with perturbation of a system, is more effective.

The devices of automatic control by the heat fluxes [1,2] intended for stabilization of the phase transition boundaries of electro-conductive liquids can be constructed, for example, on use of high-frequency electromagnetic fields. The action principle of such systems is the following: the curvature of the phase transition boundary causes change of current in the operating winding. The induced secondary current in a thin skin-layer, almost coinciding with a boundary of phase transition, owing to Joule thermal emissions stabilizes a surface of the front of solidification, etc.

Choosing the parameters of a control system [1,2] allows achieving the suppression of practically any kind instability of boundaries of phase transition that opens possibility for stabilization of a garnissage and excludes possibility of contact of metal with the channel walls and crucibles of metallurgical devices [2-4].

2.1 Control of boundaries in continua

Theoretical research of problems of the boundary control of phase transition is conducted by creation of physical and mathematical models of the corresponding physical phenomena. The models are constructed, whenever possible, simple but reflecting main features of the studied phenomenon. Complex mathematical models for the most general cases are realized on computer.

There are many works on description of various features of interaction of electromagnetic fields with liquid metals and on application of the revealed effects in technologies [3,6-9], however in metallurgy and electric welding a little attention, despite importance of this task for practice is still paid to research of features of the boundary control in phase transition. By control of the boundary form of phase transition by means of systems of automatic control [2] on the interface connected with the regulator the impedance boundary conditions are stated. And, as the regulator is, as a rule, connected to a considerable power source, the reverse influence of object on the regulator is insignificant and can be neglected that substantially simplifies a task.

2.1.1 Thin skin-layer near crystallization surface

At the boundary control of phase transition by means of high-frequency electromagnetic fields of the above-mentioned way [1,2] the impact on a boundary is thermal (Joule thermal emissions) owing to what the frequency of an electromagnetic field cannot appear in the solution of a task if connection with a control system is set by means of conditions (1). In this case control of a configuration of liquid and solid phases is based on the phenomenon that the field freely penetrates through a thin layer of a wall solid phase and quickly fades (owing to a considerable difference in conductivity of phases) in a thin skin-layer near a crystallization surface. Thus, as a first approximation, the skin-layer thickness can be neglected considering it much less than characteristic size of an ingot and assuming it coinciding with a surface of crystallization (solidification), which actually is also an area of some thickness.

2.1.2 Frequency of oscillations of the interfacial form for crystal-melt

Oscillations of the boundary of phase transition, as shown in [2-4], are low-frequency unlike fluctuations excited by fields through an action of ponderomotive forces, i.e. such systems have considerable inertness in the thermodynamic relation. Stabilization of boundary of phase transition (suppression of unstable harmonicas) happens due to Joule heat fluxes owing to what the action of a field is not direct and, as showed researches; the power expenses are rather small. It is known [10] that the interfacial form crystal-melt significantly changes depending on crystallization conditions.

Real interfacial surfaces owing to high diffusive mobility of atoms at a temperature of melting and rather low value of surface energy have an essential curvature in the scales commensurable with sizes of an elementary cell. Degree of a roughness of an interfacial surface crystal-liquid phase is defined by change of its free energy in the course of chaotic accession of atoms [10,11]. A curvature of boundaries of phase transition is a consequence of loss of stability as a result of action of the field of stresses and deformations.

2.2 Stabilization of boundaries in continua

Application of electromagnetic fields gives the chance to stabilize the deformation of boundaries of phase transition that in certain cases has paramount importance. Such oscillations of the boundaries have some regularity, which is considered here.

2.2.1 Physical model of the system

Investigation of the excitation and suppression of oscillations of the phase change boundaries of transformation from liquid to solid state is starting from the case of Eigen oscillations.

The cylindrical channel which wall can consist of any finite number of layers of various materials is considered. On an internal surface of such channel there is a thin layer of a solid phase (film) formed from a melt in the channel by creation of special temperature condition. The solid film on channel walls (garnissage), on the one hand, protects walls from thermal and other destroying influences and, on the other hand, protects melt from pollution by different impurity that is important for branches of special metallurgy in which increased requirements to purity of the melted and (or) transported liquid metal are imposed.

The block diagram of the studied physical system including a configuration from liquid and solid phases of the same material is submitted in Fig.1, where R_0 - the radius of the area occupied with melt, r_0 - thickness of a layer of a solid phase (a film, a garnissage), $r = R_0$ - a cylindrical surface of melt crystallization (the front of crystallization, border of phase transition from a liquid to solid state), $r = R_0 + r_0$ - the internal radius of the channel. The task is considered in cylindrical coordinate system $0r\varphi x$, the axis $0x$ is directed along the symmetry axis of the channel.

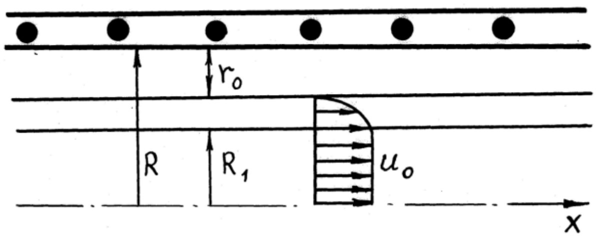

Fig.1 Cylindrical configuration liquid-solid phases with interfacial boundary from liquid to solid

Actually the form of interfacial surface crystal-melt is never expressed by smooth function as it significantly depends on conditions of crystallization [10,11]. The relief of a surface of crystallization undergoes perturbations, both in scales of the sizes of crystallites, and in scales, commensurable with area of distribution of the distortions caused by capillary forces. Mostly we consider effects of the second type, i.e. we neglect a microstructure of the phase transition boundary.

Value of surface energy on the boundary of a pure crystal and liquid phase can be determined by comparison of a structure of a pure crystal and liquid. By metal melting its specific volume increases approximately by 3% that occurs because of increase in average distance between atoms and, as a result, - increases in a potential component of internal energy. Because of more disorder structure of liquid metals in comparison with the solid phase there is a distortion of a surface crystal-melt [10,11]. Therefore real interfacial boundaries owing to high diffusive mobility of atoms at a temperature of melting and rather low value of surface energy are significantly curved in scales of the sizes of an elementary cell. However we consider processes at the macro-level when the characteristic linear size of system considerably surpasses the scale of an elementary cell and thereof the accepted structural scheme correctly reflects a state of physical system in macro-level.

Considering model of physical system according to the presented structural scheme, we believe that in an equilibrium state the surface of crystallization $r = R_0$ has the constant temperature equal to the melt crystallization temperature T_*, and conditionally we replace an area of phase transition with a surface (cylindrical area of zero thickness). Also we consider that surface is deformed only continuously, without rupture. Then the mathematical model of the described system can be presented as follows.

2.2.2 Mathematical model of the system

The mathematical model of the considered system has to include the equations of mass, impulse and energy conservation for liquid and solid phases (taking into account a channels' wall) with the corresponding boundary conditions. Generally it is the conjugated boundary task, which solution represents independent interest. We focus on possibility of the boundary control, suppression of its instability, the equilibrium state is supposed known and for the purpose of simplification of the considered tasks for its description various simple mathematical models reflecting only the main features of physical process.

Thus, if melt is immovable in the unperturbed state and the phase transformations are absent, then the equation array of mass and impulse conservation are satisfied and the energy equations is

$$\frac{\partial}{\partial r}\left(\chi_n \frac{\partial T_n}{\partial r}\right)+\frac{1}{r}\frac{\partial\left(\chi_n T_n\right)}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(\chi_n \frac{\partial T_n}{\partial \varphi}\right)+ +\frac{\partial}{\partial z}\left(\chi_n \frac{\partial T_n}{\partial z}\right)=0, \qquad (2)$$

where index n means n-th layer, $n=\overline{1,N}$, n=1 corresponds to melt, n=2- to solid film, $n=\overline{3,N}$ - the layers of the channel's wall.

The heat conductivity coefficient generally is considered variable (depends on temperature) because at small thickness of layers and big differences of temperatures dependence κ_n from T_n can affect behavior of system. Thicknesses of layers are respectively designated r_n, where $r_1=R_0, r_2=r_0$, etc. The boundary conditions for system of the differential equations (2) are stated as follows: a symmetry condition on channel axis, equality of temperatures and thermal fluxes on all boundaries of interface of layers and temperature conditions on an external surface of the channel, for example, the law of heat exchange with surrounding environment. Therefore account the above stated yields:

$$z=0, T_n=T_{n0}(r); \quad z=L, T_n=T_{nl}(r);$$

$$r=0, T_1=T_0(z); \quad r=\sum_{i=1}^{n} r_i,\ T_n=T_{n+1}, \qquad (3)$$

$$\kappa_n\, dT_n/dr=\kappa_{n+1}\, dT_{n+1}/dr; r=r_N, \kappa_N\, dT_N/dr=\lambda_{NC}\left(T_N-T_c\right),$$

where λ_{NC}, T_c are the coefficient of heat transfer for N-th layer (external) with surrounding medium and the temperature of surrounding medium, respectively.

If the system has an axial symmetry (a flow without twisting, a wall of uniform by φ layers), the equations (2) become simpler and boundary conditions (3) allow closing the corresponding boundary task. The solution of such boundary-value tasks will be given below for a number of concrete physical situations as within the linear low-amplitude theory the equations of the perturbed system are linearized relating the unperturbed (equilibrium) state described by these tasks.

2.2.3 Oscillations of the melt solidification front

Under certain conditions the described equilibrium condition of the system can be broken by casual or regular internal or external perturbations. Therefore it is interesting to consider a perturbed system. Such task within the linear low-amplitude theory can be considered as follows. We assume that in some time point all parameters of the system (temperature, pressure, boundary of the phase transition, etc.) received small perturbations in the form of progressive waves

$$q=Q(r)e^{i(kx+m\varphi-\omega t)} \qquad (4)$$

where Q - complex amplitude of perturbation.

2.2.4 Mathematical model of perturbed system

With account of the above and (2) linearization of the perturbed equation array gives

$$\rho_n c_n \frac{\partial \tau_n}{\partial t}=div(\kappa_n \nabla \tau_n), \qquad (5)$$

where τ_n, c_n - perturbations of temperature and coefficient of heat capacity for n-th layer (for simplicity in many cases it is supposed $\rho_n, c_n, \kappa_n - const$). In more complex case the influence of convection on a system's stability (spreading of small-amplitude perturbation) is taken into account, the equation array for perturbations is as follows:

$$\frac{\partial \rho_1}{\partial t}=-div\rho_1\vec{v}_1, \quad \rho_n c_n \frac{\partial \tau_n}{\partial t}=div(\kappa_n \nabla \tau_n), \qquad (6)$$

$$\frac{\partial \vec{v}_1}{\partial t}=-\frac{1}{\rho_1}\nabla p_1, \quad \rho_1 c_1\left(\frac{\partial \tau_1}{\partial t}+\vec{v}_1\nabla T_1\right)=div\left(\kappa_1 \nabla \tau_1\right),$$

where T_1 - temperature distribution for equilibrium system, $\vec{v}_1, \rho_1$ - perturbations of flow velocity and pressure, $n = \overline{2, N}$. Coefficients of heat diffusivity are considered as constants and melt inviscid. Further influence of melt viscosity on development of perturbations of system will be shown, here it is important to be limited to the remark that at many metal melts viscosity is rather low (for example, at cast iron the kinematic coefficient of viscosity is less, than at water) and dissipative processes don't play an essential role.

Let us analyze consecutively the systems (5) and (6). The boundary conditions for (5) include as subset the boundary conditions for (3), therefore:

- on the interfacial boundary of phase change stated as progressive wave $r = R_0\left[1+\zeta e^{i(kx+m\varphi-\omega t)}\right]$ the temperature perturbation is zero, therefore with accuracy to linear terms follows

$$\tau_j\left(R_0,\varphi,x\right)+R_0\zeta\left(\frac{\partial T_j}{\partial r}\right)_{r=R_0} e^{i(kx+m\varphi-\omega t)}=0, \qquad (7)$$

where j=1,2; on the same surface the conditions of mass and heat conservation yield at $r = R_0$,

$$u=\left(1-\rho_{21}\right)\frac{\partial r}{\partial t},\ \kappa_2\frac{\partial\tau_2}{\partial r}=\kappa_1\frac{\partial\tau_1}{\partial r}+\rho_2\lambda_{21}\frac{\partial r}{\partial t}, \qquad (8)$$

where $\partial r/\partial t$ - speed of the front crystallization, $\rho_{21}=\rho_2/\rho_1$, λ_{21} - specific heat of melting;

- on the axis of channel from the symmetry condition, absence of the velocity and temperature perturbations results

$$r=0,\quad u=0,\quad T_1=0; \qquad (9)$$

- on the boundaries of solid layers the conditions of equal temperatures and heat fluxes yield

$$r=\sum_{j=1}^{n} r_j;\quad \tau_n=\tau_{n+1},\quad \kappa_n\frac{d\tau_n}{dr}=\kappa_{n+1}\frac{d\tau_{n+1}}{dr}; \qquad (10)$$

- on the external surface of the channel, which may be thermo-isolated or performing heat exchange with surrounding (and in case of system of heat control it may be automatically controlled) the following condition is stated

$$r=\sum_{j=1}^{N} r_j;\quad \kappa_N\frac{d\tau_N}{dr}=-G_{k,m}\tau_N. \qquad (11)$$

Here $G_{k,m}$ may be 0, λ_{NC} or equal to the coefficient of the feedback control system [2] for the three above considered cases. The set of the equations and correlations (4)–(11) represents the mathematical model for considered physical system. To reveal the dimensionless criteria determining the processes, as well as for generalization of the model, it is useful to transform it to dimensionless form.

2.3 Dimensionless form of the model

Introduced as the scales for the length, time, frequency, velocity, pressure, density and temperature, respectively R_0, R_0/a_{1*}^2, a_{1*}^2/R_0^2, a_{1*}^2/R_0, $\rho_{1*}a_{1*}^4/R_0^2$, ρ_{1*} and T_*, the equation array (4), (6) transforms to the dimensionless form:

$$\bar{q}=\bar{Q}\left(\bar{r}\right)e^{i\left(\bar{k}\bar{x}+m\varphi-\bar{\omega}F_0\right)},\ \frac{\partial\bar{\rho}_1}{\partial\,\mathrm{Fo}}=-div\bar{\rho}_1\vec{\bar{v}}_1,\ \frac{\partial\vec{\bar{v}}_1}{\partial\,\mathrm{Fo}}=-\frac{1}{\bar{\rho}_1}\nabla\bar{p}_1,$$

$$\frac{\partial\bar{\tau}_1}{\partial\,\mathrm{Fo}}+\vec{\bar{v}}_1\nabla\bar{T}_1=\frac{div\left(\bar{\kappa}_1\nabla\bar{\tau}_1\right)}{\bar{\rho}_1\bar{c}_1},\ \frac{\partial\bar{\tau}_n}{\partial\,\mathrm{Fo}}=\frac{div\left(\bar{\kappa}_n\nabla\bar{\tau}_n\right)}{\bar{\rho}_n\bar{c}_n}, \qquad (12)$$

where $\mathrm{Fo}=a_{1*}^2 t/R_0^2$ - Fourier number (ratio of characteristic time of the process to the relaxation time of temperature perturbations),

$$\bar{\rho}_n=\rho_n/\rho_{1*},\ \bar{c}_n=c_n/c_{1*},\ \bar{\kappa}_n=\kappa_n/\kappa_{1*},\ a_{1*}^2=\frac{\kappa_{1*}}{\rho_{1*}c_{1*}},$$

where hyphens mean dimensionless values, and asterisks - belonging to a condition of phase transition. Melt is supposed almost incompressible owing to what heat of expansion (compression) can be neglected. Further, the same as earlier, we omit hyphens at dimensionless sizes for simplicity.

The boundary conditions (7)-(11) are:

$$r=1,\quad \tau_j+\zeta\frac{\partial T_j}{\partial r}e^{i(kx+m\varphi-\omega F_0)}=0,\quad j=1,2,$$

$$u=\left(1-\rho_{21}\right)\frac{\partial r}{\partial\,\mathrm{Fo}},\ \kappa_{21}\frac{\partial\tau_2}{\partial r}=\frac{\partial\tau_1}{\partial r}+\frac{R_\lambda}{\kappa_1}\frac{\partial r}{\partial\,\mathrm{Fo}}; \qquad (13)$$

$$r=0,\quad u=0,\quad \tau_1=0; \qquad (14)$$

$$r=\sum_{j=1}^{n} r_j,\ \tau_n=\tau_{n+1},\ \frac{d\tau_n}{dr}=\frac{\kappa_{n+1}}{\kappa_n}\frac{d\tau_{n+1}}{dr};$$

$$r=\sum_{j=1}^{N} r_j,\ \frac{d\tau_N}{dr}=-Bi_{k,m}\tau_N. \qquad (15)$$

where $r=1+\zeta e^{i(kx+m\varphi-\omega F_0)}$ is a phase change surface. Here $R_\lambda=\frac{\rho_2\lambda_{21}}{\rho_{1*}c_{1*}T_*}$ - dimensionless criterion characterizing ratio of melting heat to heat capacity at the boundary of phase change, $Bi_{k,m}=Q_{k,m}R_0/\kappa_N$ - modified Biot number. At $Q_{k,m}=\lambda_{NC}$ (external surface of the channel is doing heat exchange with surrounding medium) $Bi_{k,m}=Bi$ and it becomes conventional Biot number determining the ratio of external and internal thermal resistances in a resting medium with a known intensity of heat transfer on a boundary. The modified Biot criterion characterizes a ratio of conductive and convective thermal resistance of boundary of an external surface of the channel with the heat control system. As in the latter case $Q_{k,m}$ can be practically any real number, $Bi_{k,m}$ can also accept any valid values owing to what this dimensionless criterion is a measure of influence of the heat flux control system [2] on physical system.

2.3.1 The parametric oscillations of the system

Considering the case of negligibly small influence of convection on oscillations of the boundary of phase transition, from the last equation (12) follows the modified Bessel equation array

$$\frac{d^2\theta_n}{dr^2}+\frac{1}{r}\frac{d\theta_n}{dr}=\left(\alpha_n^2+\frac{m^2}{r^2}\right)\theta_n, \qquad (16)$$

where $\alpha_n^2=k^2-ib_n\omega$, $b_n=R_0^2/a_n^2$, and θ_n is dimensionless amplitude of the temperature perturbation for *n*-th layer. Boundary conditions (13)-(15) for this case transform to the form [9]:

$$\theta_n(s_n)=\theta_{n+1}(s_n), \quad \theta_n^{'}(s_n)=\kappa_{n+1}/\kappa_n\theta_{n+1}^{'}(s_n),$$

$$\theta_1(1)=-1, \quad \theta_N^{'}(s_N)=-Bi_{k,m}\theta_N(s_N), \qquad (17)$$

where $n=\overline{2,N-1}$, dash means derivative by r, $s_n=1+\sum_{j=2}^{n}r_j$. The coefficients are assumed constant.

General solution of the equations (16) is:

$$\theta_n=c_{2n-1}K_m(\alpha_n r)\theta_n+c_{2n}I_m(\alpha_n r), \qquad (18)$$

where K_m, I_m- the modified Hankel and Bessel functions of *m*-th order, c_j ($j=\overline{1,2N}$)- constants determined from substitution of (18) into (17).

2.3.2 Integral correlations

The following artificial approach is used to compute the Eigen values $\lambda=i\omega$ and get the properties of the Eigen functions without solving the dispersive equations. Multiplying the equations (16) by $r\theta_n^*$, where θ_n^*- complex conjugated function with θ_n, and then integrating the equation obtained by r from s_{n-1} to s_n ($n=\overline{2,N}$), with $s_1=1$:

$$\lambda b_n\int_{s_{n-1}}^{s_n}|\theta_n|^2 rdr=\int_{s_{n-1}}^{s_n}\left(k^2+\frac{m^2}{r^2}\right)|\theta_n|^2 rdr+\int_{s_{n-1}}^{s_n}\left|\theta_n^{'}\right|^2 rdr+$$
$$+s_{n-1}\theta_n^*(s_{n-1})\theta_n^{'}(s_{n-1})-s_n\theta_n^*(s_n)\theta_n^{'}(s_n). \qquad (19)$$

The equation (19) thus obtained may have more solutions than (18) due to integration. But if this equation gives stability against small-amplitude perturbations (fading of oscillations with time) then the system described by (18) is stable too. Thus, the stability conditions obtained are necessary but not sufficient whereas the linear theory of perturbations gives only sufficient conditions of instability.

Therefore, if the system at such approach is steady, it especially will be steady at the solution (18) with the subsequent definition of Eigen numbers of a task. It is only necessary to notice that stability in relation to low-amplitude perturbations (even strong stability in the above described sense) doesn't mean yet stability of physical system, which can be unstable in relation to perturbations of finite amplitude.

2.3.3 Eigen values of interfacial oscillations

Multiplying further the left and right parts of the equations (19) on κ_n and summing the equations obtained by all $n=\overline{2,N}$, with account of the boundary conditions (17) yields

$$\lambda=\frac{1}{q_2}\left[q_1-\kappa_1\theta_n^{'}(1)+\kappa_N s_N\left|\theta_N(s_N)\right|^2 Bi_{k,m}\right], \qquad (20)$$

where

$$q_1=\sum_{n=2}^{N}\kappa_n\int_{s_{n-1}}^{s_n}\left[\left(k^2+\frac{m^2}{r^2}\right)|\theta_n|^2+\left|\theta_n^{'}\right|^2\right]rdr,$$
$$q_2=\sum_{n=2}^{N}b_n\kappa_n\int_{s_{n-1}}^{s_n}|\theta_n|^2 rdr, \qquad (21)$$

Oscillations of the parameters are fading in time if a real part of the Eigen values λ is positive. Therefore the necessary stability condition for a system with account of (20), (21) is written as

$$re\left[\theta_1^{'}(1)\right]\le q_1/\kappa_1+s_N\bar{\kappa}_N\left|\theta_N(s_N)\right|^2 Bi_{k,m}, \qquad (22)$$

where the equal-sign corresponds curve loss of stability (the neutral curve dividing areas of steady and unstable conditions of physical system). The sufficient condition of instability turns out by change of a sign of inequality (22) to the opposite one. As it is easily noted, from the necessary stability (against infinitesimal perturbations) condition (22), the temperature distribution in an equilibrium state and amplitude of perturbation of the interface of phase transition don't influence stability of the system in the considered model statement. Stability of the system, irrespective of amplitude of small perturbations, is defined only by physical parameters of a multilayered wall of the channel and a type of oscillations.

The system is always steady at $re\theta_1^{'}(1)\le 0$ as in this case in the left part of an inequality (22) there is a negative value whereas in right part (see (21)) - positive. Physically this case corresponds to decrease of amplitude of a temperature perturbation in the direction to an external surface of the channel (perturbations arise inside the channel and fade in a wall). Thus, it follows that increase of parametric oscillations of system in time (instability) can take place only in the presence of external perturbations. If reliable thermal insulation of the channel outside is possible, any parametric oscillations will fade eventually. If it isn't possible to exclude external perturbations, for research of stability it is necessary to solve the inequality (22), which allows defining values of coefficient of the feedback control system for heat flux [2] intended for suppression of those harmonicas promoting instability development.

3 The mutual influence of system's parameters and stability condition

For electromagnetic suppression of perturbations the modified Biot number $Bi_{k,m}$ (for each mode with the wave numbers k,m) must be selected from the condition (22) that is always available because $Bi_{k,m}$ can get any real value.

On the basis of the considered mathematical model it is possible to carry out analysis of stability of the cylindrical front of melt crystallization, which is in an equilibrium motionless state in the multilayered channel. So, for $N=3$ (the single-layer channel) taking into account the above, from (17), (18) follows:

$$c_j=(-1)^j\delta_j/\delta_0,\ \left(j=\overline{1,6}\right),\quad \delta_0=\varepsilon_6 Bi_{k,m}-\varepsilon_5,$$
$$\delta_j=\varepsilon_{2j+6}Bi_{k,m}-\varepsilon_{2j+5},\ K_m=A_1,\ I_m=A_2,\ z_1=\alpha_1 s_1,$$
$$z_2=\alpha_1,\ z_3=\alpha_3 s_3,\ z_4=\alpha_3 s_2,\ z_5=\alpha_2 s_2,\qquad (23)$$
$$\varepsilon_1=\mu_1' A_2'(z_6)-\mu_2' A_1'(z_6),\ \ \varepsilon_{2p+13,2p+14}=\mu_{13}' B_{1,0}(z_3),$$
$$\varepsilon_2=\mu_1' A_1(z_6)+\bar{\kappa}_2-\mu_2' A_1'(z_6),\ \ \mu_{13}'=\prod_{q=1}^{2}\bar{\kappa}_q\mu_{1q}',$$
$$\varepsilon_{2p+5,2p+6}=\sum_{q=0}^{1}\bar{\kappa}_{q+1}B_{1-q}(z_1)\sum_{g=0}^{1}\frac{\kappa_{g+2}}{\kappa_{g+1}}\mu_{6-2g,5-2g}'\mu_{10-2q-g}',$$
$$\varepsilon_{2p+9,2p+10}=\mu_{11}'\sum_{q=0}^{1}\frac{\kappa_{q+2}}{\kappa_{q+1}}\mu_{6-2q,5-2q}'B_{1-q}(z_5),\ \ A_p=A_0,$$
$$\varepsilon_{5,6}=\sum_{q=0}^{1}\frac{\kappa_{q+2}}{\kappa_{q+1}}\varepsilon_{4-q}\mu_{6-2q,5-2q}',\quad \varepsilon_{3,4}=\sum_{p=1}^{2}\varepsilon_p B_{0,1}(z_5),$$
$$\mu_{4p}'=\sum_{q=0}^{1}(-1)^q A_1'\left(z_{2p+q+1}\right)A_2'\left(z_{2p-q+2}\right),\ (p=1,2),$$
$$A_p'=B_1,\quad \mu_{2,11}=A_1(z_{2,1})A_2'(z_1)-A_1'(z_1)A_2(z_{2,1}).$$

And then μ_{4u-3}' is obtained from μ_{4p} by replacing A_1',A_2' for A_1,A_2 and z_{2p+q+1},z_{2p-q+2} for z_{2p-q},z_{2p+q-1}, where u=1,2,3, and $\mu_{3,6}'$ is got from $\mu_{2,11}'$ with substitution $z_{2,1},z_1$ on $z_{3,4},z_{4,3}$ and $\mu_{7,10,12}'$ on μ_2' substituting $z_1=z_{6,5,5}$, $z_2=z_{5,6,5}$. Here for the purpose of compactness the conditional system of symbolic representation of formulas is entered: all indexes with commas mean that the considered expression breaks up to some expressions and indexes of the variables in them accept consecutively those values, which are listed in a line through a comma.

3.1 Stability of the interface between solid and liquid phases

The stability condition for physical system (including boundary of phase transition) (22) with account (23) and (18) allows obtaining for the case considered above the following condition:

$$\sum_{n=1}^{3}\bar{\kappa}_n\int_{s_{n-1}}^{s_n}\sum_{q=1}^{2}\left(k^2+\frac{m^2}{r^2}\right)^{q-1}\left|\sum_{p=1}^{2}(-1)^p\left(\varepsilon_{4n+2p+2}Bi_{k,m}-\varepsilon_{4n+2p+1}\right)\right|^2\cdot$$
$$\cdot B_{2-q}^2(\alpha_n r)rdr+\qquad (24)$$
$$+\bar{\kappa}_3 s_3\left|\sum_{p=1}^{2}(-1)^p\left(\varepsilon_{2p+14}Bi_{k,m}-\varepsilon_{2p+13}\right)B_0(\alpha_3 s_3)\right|^2 Bi_{k,m}\geq$$
$$\geq re\left[\left(\varepsilon_6 Bi_{k,m}-\varepsilon_5\right)^{-1}\sum_{p=1}^{2}(-1)^p\left(\varepsilon_{2p+6}Bi_{k,m}-\varepsilon_{2p+5}\right)B_1(\alpha_1)\right]\cdot$$
$$\cdot\left|\varepsilon_6 Bi_{k,m}-\varepsilon_5\right|^2,$$

where re, as before, means real part of complex value. From here it is visible that at increase of oscillations of system in time they can be suppressed by means of high-frequency electromagnetic fields and thermal control systems. For this purpose it is necessary to create the special thermal mode on an external surface of the channel and automatically support it. Really, supposed $Bi_{k,m}\gg 1$, one can get from (24) the following

$$\sum_{n=1}^{3}\bar{\kappa}_n\int_{s_{n-1}}^{s_n}\sum_{q=1}^{2}\left(k^2+\frac{m^2}{r^2}\right)^{q-1}\left|\sum_{p=1}^{2}(-1)^p\varepsilon_{4n+2p+2}B_{2-q}(\alpha_n r)\right|^2 rdr+$$
$$+\bar{\kappa}_3 s_3\left|\sum_{p=1}^{2}(-1)^p\varepsilon_{2p+14}B_0(\alpha_3 s_3)\right|^2 Bi_{k,m}\geq\geq\qquad (25)$$
$$\geq\geq\left|\varepsilon_6\right|^2 re\left[1/\varepsilon_6\sum_{p=1}^{2}(-1)^p\varepsilon_{2p+6}B_1(\alpha_1)\right].$$

It is easy to notice that the condition (25) can be always satisfied with selection of the modified Biot number $Bi_{k,m}$ (for each pair k,m). Thus intensity of increase (fading) of oscillations in time is defined by value λ. The criterion of a thermal homochromatic Fo characterizes similar temporary pictures of oscillations' development.

3.1.1 Selection of the Biot number

Selecting the parameter $Bi_{k,m}$ allow controlling the fading rate (by $Bi_{k,m} > Bi^*_{k,m}$) or increasing rate (by $Bi_{k,m} < Bi^*_{k,m}$) of perturbation of the boundary phase transition. Here $Bi^*_{k,m}$- critical value of the Biot number $Bi_{k,m}$ for the given mode with the wave numbers k,m. With choosing value $Bi_{k,m}$ individually for each pair $k,m,$ the high resolution of the control system by modes' of oscillations is available.

In specific case of thermo-isolated channel, $Bi_{k,m}=0$, therefore from (24) follows

$$|\varepsilon_5|^2 re\left[\frac{1}{\varepsilon_5}\sum_{p=1}^{2}(-1)^p \varepsilon_{2p+5}B_1(\alpha_1)\right] \le \qquad (26)$$

$$\le \sum_{n=1}^{3}\bar{\kappa}_n \int_{s_{n-1}}^{s_n}\sum_{q=1}^{2}\left(k^2+\frac{m^2}{r^2}\right)^{q-1}\left|\sum_{p=1}^{2}(-1)^p \varepsilon_{4n+2p+1}B_{2-q}(\alpha_n r)\right|^2 rdr,$$

where from it is seen that increase of the heat conductivity coefficients of the channel's layer ($\bar{\kappa}_n$) and their thickness ($\bar{r}_n$) improves stability of the system, e.g. decreases the critical modified Biot number $Bi^*_{k,m}$.

3.1.2 Short-wave perturbations of the system

By $k \gg 1$ yields $\alpha_n \approx k$, therefore from (26), (24), accounting asymptotic behaviors of the modified Bessel and Hankel functions, the following simple condition of the oscillations' growing yields:

$$\beta k e^{-k} \le e^{2ks_3} - e^{2ks_2} + \beta\frac{m^2}{k}e^{-2ks_3}\sum_{n=1}^{3}\bar{\kappa}_n\int_{s_{n-1}}^{s_n}|\theta_n|^2\frac{dr}{r},$$

where $\beta = s_2\sqrt{s_1}\frac{\pi}{16}\left[\bar{\kappa}_2 s_1 + \bar{\kappa}_3 s_2 + \left(\bar{\kappa}_3+\bar{\kappa}_2^2\right)s_1^{n-1}s_2\right]$. It shows that short-wave fluctuations of boundary of phase transition can accrue in time only to a certain value k, at which excess it becomes impossible.

Excitation of short-wave oscillations by means of external influences [9] in this case demands considerable expenses of energy, influence is not power here, but thermal one, low-frequency and a low-depending on the frequency of an external high-frequency field [2]. Intensity of this influence is determined by dimensionless criterion $Bi_{k,m}$.

3.1.3 Long-wave perturbations of the system

By $k \ll 1$ put in (18) $\lambda = 0$ ($k = \pm\alpha_n$), then from (26) follow neutral stability curve, which is boundary for transition of the system from stable to unstable state:

$$\sum_{n=1}^{3}\bar{\kappa}_n\left(s_n^{2m}-s_{n-1}^{2m}\right) = \frac{2m^2k}{2m^2+0,5}\left(\frac{1}{m}+\bar{\kappa}_2 s_{21}\right), \qquad (27)$$

where is $m = 1,2,3,...$, $s_{21} = s_2/s_1$.

The oscillations in the system satisfying the condition (27) are spreading with a constant in time amplitude. Analyzing the condition (27), one can see that oscillations of constant amplitude by high values of the wave number m (by φ) and thickness of the layers of channel wall $\bar{r}_n$ are impossible in reality because the value to the left in (27) substantially prevails the value to the right in this equation. The physical sense of this conclusion consists that short waves by azimuthally (φ) coordinate (twisting of a flow and all system with crystallization boundary) demand huge power expenses. Here m- integers as in the cylindrical channel the wave has to become closed by φ. Influence of the accepted simplifications affects that the semi-infinite channel on an axis x is an impediment in wave spreading by φ as far as the infinity "extinguishes" perturbations.

By comparably small m and r determining by length of longitudinal waves the twisted oscillations (by coordinate φ) are available. In case of axisymmetrical oscillations ($m=0$) follows $\theta_1'(1) = -\bar{\kappa}_2\prod_{n=1}^{3}s_n < 0$ and thus fading of perturbations happens with time (the system is stable).

Thus, it is revealed that short-wave longitudinal perturbations of system, irrespective of character of a twisting, fade in time if temperature of an external surface of the channel is maintained by a constant (reliable thermal insulation of the channel). The most real for practice long-wave perturbations of system parameters can have both the constant and increasing in time amplitude. It is determined by length of waves by coordinate φ.

Most often in practice it is necessary to suppress the growing fluctuations of parameters in technological processes. The considered model tasks show that it can be carried out, irrespective of physical conditions in which technological process is running, by a statement of specially selected mode of heat exchange on an external surface of the channel controlled by automatically operated high-frequency electromagnetic field [2-4].

3.2 Influence of the melt's convection on stability of the crystallization front

In many metallurgical facilities the stability of the boundary of melt's front crystallization is important as quality of technological process can significantly depend on it. The same task is important for solution of the problem for the channel walls' and lining protection for metallurgical units against thermal and chemical destruction by means of an artificial garnissage [4,9].

However garnissage as a protecting method is effective only in case of possibility of automatic control of the boundaries of crystallization and suppression of their instability. Therefore it is necessary to conduct research of convection influence on stability of the front of crystallization and possibility of its stabilization by means of the directed external influence in case of instability. For this purpose we will analyze system of the differential equations (6), considering for simplicity gradients of density and heat conductivity small due to negligible small considered perturbations of temperature, which functions these physical characteristics are. This simplification is physically correct by weak dependence of density and heat conductivity on temperature.

3.2.1 Model thin area of constant temperature

If influence of the channel wall is neglected, then without account of convective terms in the energy conservation equation for melt, the earlier considered case takes a place. But the boundary conditions are different [4]:

$$\theta_1(s_0)=0;\quad \theta_1(1)=-\zeta\ln s_0,\quad \theta_2(1)=-\frac{\kappa_{12}}{\ln s_0}\zeta; \qquad (28)$$
$$\kappa_{21}\theta_2'(1)=\theta_1'(1)-\lambda R_\lambda\zeta;\quad \theta_2'(s)=-Bi_{k,m}\theta_2(s),$$

where $\kappa_{12}=\kappa_1/\kappa_2$ by r =1. $s=1+r_0/R_0$. Here for elimination of mathematical feature of the solution (18) at r =0 (temperature addresses in infinity if coefficient at K_m is nonzero, and otherwise temperature addresses in zero, as $I_m\neq 0$ only at m =0), the area close to axis with a constant temperature is entered. This area models some kind of power source and its dimensionless radius is stated s_0 (some kind a core or a narrow channel in the main channel creating a surface of constant temperature T_0).

As shown in [3], the value R_λ for crystalline solids is rather big that gives the chance from the analysis of a condition of phase transition (28) to conclude the following. First, the Eigen values λ of a task are small as otherwise on the boundary of phase transition there would have to be physically unreal big gradients of temperature. Secondly, as the right part of (28) contains a big value R_λ, then searching the solution in a form of series by a small parameter λ, it is possible to compute λ already in a zero approach from (28). From this follows that fluctuations of boundaries of phase transition are low-frequency (thermal processes have big characteristic time of perturbations' development), whereas it is convenient to apply high-frequency electromagnetic fields [2,3] to their stabilization.

If use the same artificial approach as applied before, it is available to show that the Eigen values of the task are real and positive. Thus, it yields:

$$\lambda=\sum_{n=1}^{2}\frac{c_{n2}\kappa_{21}^{2n-2}\ln s_0}{\left(c_{n2}\kappa_{21}^2+c_{11}\right)\ln s_0-\zeta R_\lambda}, \qquad (29)$$

where

$$c_{n1}=\int_a^b \bar{a}_n^2\left|\theta_n\right|^2 r dr, \qquad (30)$$
$$c_{n2}=\int_a^b\left[\left(k^2+\frac{m^2}{r^2}\right)\left|\theta_n\right|^2+\left|\theta_n'\right|^2\right]rdr+s(n-1)Bi_{k,m}\left|\theta_2(s)\right|^2.$$

Here $\bar{a}_n^2=a_n^2/a_{1*}^2$, $a=s_0, b=1$ by $n=1$ and $a=1, b=s$ by $n=2$. Apparently from (29), (30), the cylindrical front of crystallization, unlike flat one [3], is steady in the considered model statement with existence of thin axial area of constant temperature (irrespective of the radius of this area, up to zero).

3.2.2 Influence of convection on stability

We estimate further an influence of convection at variable physical parameters of system. For this purpose put density function of temperature $\rho(T)$ and expand it in a Taylor series in the vicinity of a point $T=T_n$, designating strokes derivatives by T:

$$\rho_n(T_n+\tau_n)=\rho_n(T_n)+\rho_n'(T_n)\tau_n+1/2\rho_n''(T_n)\tau_n^2+...,$$

where from seen that $\partial\rho_n/\partial t$ can be considered as a small value of higher order comparing to $\nabla\rho_n$, as far as

$$\frac{\partial\rho_n}{\partial t}=\rho_n'(T_n)\frac{\partial\tau_n}{\partial t}+\rho_n''(T_n)\tau_n\frac{\partial\tau_n}{\partial t}+\frac{\rho_n'''}{2}(T_n)\tau_n^2\frac{\partial\tau_n}{\partial t}+...,$$

where $\rho_n'(T_n)=\left(\partial\rho_n/\partial T\right)_{T=T_n}$.

With account of the above and (11), the system (12) results in a linear approach

$$u_1=\frac{-1}{\bar{\rho}_1\lambda}\frac{dp_1}{dr},\ v_1=\frac{mp_1}{\bar{\rho}_1\omega r},\ w_1=\frac{kp_1}{\bar{\rho}_1\omega},\ \frac{du_1}{dr}+i\left(m\frac{v_1}{r}+kw_1\right)+$$
$$+\frac{u_1}{r}=0,\quad \frac{d^2\theta_n}{dr^2}+\frac{1}{r}\frac{d\theta_n}{dr}-\left(\delta_n^2+\frac{m^2}{r^2}\right)\theta_n= \qquad (31)$$

$$=\frac{2-n}{\bar{a}_n^2}\left[\frac{1}{\lambda}\frac{\partial T_1}{\partial r}\frac{\partial p_1}{\partial r}+\frac{p_1}{\omega}\left(\frac{m}{r^2}\frac{\partial T_1}{\partial \varphi}+k\frac{\partial T_1}{\partial x}\right)\right].$$

Here $\{u_1, v_1, w_1\}, p_1$- amplitudes of perturbations of velocity and pressure, $\delta_n^2 = k^2 - \lambda / a_n^2$.

From the first four equations of the system (31) yields the modified Bessel equation for pressure perturbation:

$$\frac{d^2 p_1}{dr^2}+\frac{1}{r}\frac{dp_1}{dr}-\left(k^2+\frac{m^2}{r^2}\right)p_1=0,$$

which has the following solution

$$p_1 = A_1 I_m(kr) + A_2 K_m(kr), \qquad (32)$$

where $A = const$. The boundary conditions (28) with account (32) and conditions of unperturbed distributed along the axis heat source ($r \le s_0$) [4]:

$$r = s_0,\ \vec{v}=0;\quad r=1,\ \frac{dp_1}{dr}=\left(1-\rho_{21}\right)\bar{\rho}_1\omega\zeta, \qquad (33)$$

which result in a system of boundary conditions for differential equations (31). From (32)-(33) is got:

$$A_1 I_m(ks_0)+A_2 K_m(ks_0)=0,\ A_1 I_m'(ks_0)+A_2 K_m'(ks_0)=0,$$

$$A_1 = A_2 = 0,\ I_m(ks_0)K_m'(ks_0)\ne I_m'(ks_0)K_m(ks_0).$$

As the modified Bessel and Hankel functions have only positive values, and the first of them is monotonously increasing, the second - monotonously decreasing, then at all values of argument other than zero, this inequality is incorrect only at $k=0, m\ne 0$. However in the latter case, owing to properties of I_m, K_m, obviously $A_1 = A_2 = 0$ and $p_1 \equiv 0$, as well. The system is steady against rather small perturbations of its parameters, i.e. parametric oscillations fade in time here.

The boundary condition can be replaced with more general: $u_1(s_0)=0$. Two other velocity components of the unperturbed melt can be other than zero (for example, in a case when the entered axial area of constant temperature isn't motionless). Then unlike considered above it turns out: $A_1 I_m'(ks_0)+A_2 K_m'(ks_0)=0,\quad A_1 I_m'(k)+A_2 K_m'(k)=\alpha\omega^2$, where is $\alpha=(1-\rho_{21})\bar{\rho}\zeta<\zeta$. As shown above, ω is small. The value α is small too for small-amplitude perturbations, therefore in a linear approach value $\alpha\omega^2$ can be neglected, where from:

$$A_1 I_m'(ks_0)+A_2 K_m'(ks_0)=0,\ A_1 I_m'(k)+A_2 K_m'(k)=0.$$

Investigation of the algebraic equation array (AEA) of two equations shows that its solution is non-trivial in case $s_0 = 1$ or $k=0$. But both cases are physically unreal because by $s_0 = 1$ all channel is filled with a melt of constant temperature $T_0 = T_*$, and by $k=0$ it is $p_1 \equiv 0$, e.g. the system is unperturbed (stability).

4 Conclusion

The model for solidification boundary interface has been developed. An analysis shown that the front of melt crystallization under neglect influence of the channel wall (the metallurgical unit) is steady against rather low-amplitude perturbations of parameters of physical system if only in an equilibrium state a melt was immovable.

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