# Discontinouity modelling for finite difference method

MLADEN MEŠTROVIĆ
University of Zagreb
Faculty of Civil Engineering
Kačićeva 26, 10000 Zagreb
CROATIA
mestar@grad.hr

*Abstract:* The finite difference metod (FDM) is widely used numerical method for numerical computation of different physical problems. The method is very useful for time-dependent problems. The sequence of finite difference method solutions with increasing the number of discrete points converge to analytical solutions. It converges rapidly for smooth problems. The simple structural engineering problem with concentrated force has discontinouity in third derivative. The problem is how to model that kind of loading for finite difference method. New approach is explained. The concentrated force is not taken as loading. It is taken as the difference of third derivative (difference between value very close left from value very close right) at loading point. The procedure for numerical evaluation is described. The presented numerical example shows that the sequence of numerical results converges with quadratic order of convergence to anaytical solution.

*Key–Words:* discontinouity, finite difference method, beam bending, concentrated force

## 1 Introduction

The finite difference method (FDM) is widely used for numerical computation of different physical problems. The method is based on the discretization of the domain on the points. The simple structural engineering problem with concentrated force has discontinouity in third derivative. The beam bedning problem is fourth-order boundary value problem. Finite difference scheme is calculated and applied for fourth derivative of displacement function. Concentrated force as function is discontinous function over beam. The function is equal to force value at the point of loading but zero at all other beam points. Applying the standard FDM scheme with loading on the right side of equation doesn't lead to solution that converges. Even the measure unit doesn't fit. The concentrated force is taken into account according its physical influence on the beam. The concentrated force gives discontinouity in transverse force at the loading point. It means that third derivative of deflection has dicontinouity at the loading point. The concentrated force is not taken as usual as loading. It is taken as discontinouity in third derivative at loading point. The third derivative of deflection function very close left to loading point is not same as the third derivative of deflection function very close right to loading point. The difference is equal to given concentrated force. It has been shown how to make numerical model to get numerical results that converge to analytical solution. The procedure is shown on simple numerical example. The third derivative is approximated only with function values at the points on the same side of the loading point. The sequence of numerical results converges to analytical result with quadratic order of convergence.

## 2 Finite difference approximation

The finite difference method approximate the derivatives with function values at discrete points. Let $w(x)$ represent a function of one variable. The function is assumed to be smooth that it has its derivative after differentiation of several times over an interval containing a particular point $\overline{x}$. The needed $n$-th derivative at point $\overline{x}$ is approximate as linear combination of $n+1$ points of the domain. The error of approximation is of order equal to $n+1$. The different choice of points taken for linear combination leads to different absolute error but the order is same. The central difference (chosen points are symmetrical about the point of interest) gives better approximation than some accidently chosen points from domain. The standard approximation, with equidistant mesh size equal to $h$ and with $w_i = w(x_i)$ and $w_{i+t} = w(x_i + t \cdot h)$, for first derivative at $x_i$ is

$$w'(x_i) \approx \frac{w_{i+1} - w_{i-1}}{2h} , \tag{1}$$

for second derivative at $x_i$

$$w''(x_i) \approx \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} , \quad (2)$$

for third derivative at $x_i$

$$w'''(x_i) \approx \frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3} , \quad (3)$$

and for fourth derivative at $x_i$ needed in beam bending problem is

$$w''''(x_i) \approx \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4} . \quad (4)$$

## 3 FDM for beam bending problem

The governing differential equation for beam bending problem, for the beam of length $L$ with flexural rigidity $EI$ under loading function $q(x)$, is given as

$$\left(EIw''\right)'' = q(x) , \quad (5)$$

with four boundary conditions. With constant flexural rigidity equation is given in form

$$EIw'''' = q(x) . \quad (6)$$

Let we divide the beam of length $L$ into $n$ equidistant parts of length $L/n$, with points ordinate expressed as

$$x_i = \frac{iL}{n} , i = 0, \ldots, n \quad . \quad (7)$$

The needed fourth derivative is expressed at any point of the beam, according the equation (4), with

$$w''''(x) \approx \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4} , \quad i = 0, \ldots, n \quad . \quad (8)$$

## 4 Boundary conditions in FDM for beam bending problem

If we use FDM scheme for fourth derivative in first two or last two points of the beam, we have unknown function values for some points outside the beam, $(w_{-2}, w_{-1}, w_{n+1}, w_{n+2})$. With introducing the boundary conditions in FDM scheme we can avoid this unknown values outside the beam.

With explicit predefined boundary values (for simply supported or clamped edge), we are going to avoid values for $w_{-2}$ and $w_{n+2}$. We do not need to take FDM scheme in that boundary points. For example at $x_0 = 0$, with given $w_0 = 0$, we loose unknown value $w_{-2}$. Similar, with $w_n = 0$ at $x_n = L$, there is no more need for $w_{n+2}$.

For simply supported edge, moment is equal to zero. The expression for moment at any point is given with

$$M(x) = -EIw'' . \quad (9)$$

The FDM scheme for second derivative leads to

$$\begin{aligned} M(0) &= -EI\frac{w_1 - 2w_0 + w_{-1}}{h^2} = 0 \\ &\Rightarrow \quad w_{-1} = -w_1 . \end{aligned} \quad (10)$$

So, we do not have $w_{-1}$ further as unknown value. Geometrically, second derivative is equal to zero at inflexion points what we can described in Figure 1 .

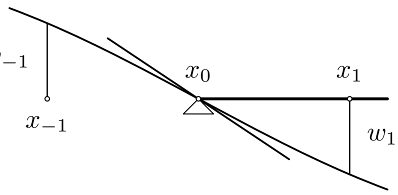


Figure 1: Substitution for unknown value at simply supported edge

For clamped edge, beam deflection (angle) $\varphi(x) = -w(x)$, is equal to zero. The FDM scheme for first derivative leads to

$$\begin{aligned} \varphi(0) &= -\frac{w_1 - w_{-1}}{2h} = 0 \\ &\Rightarrow \quad w_{-1} = w_1 . \end{aligned} \quad (11)$$

Now we do not have $w_{-1}$ further as unknown value. Geometrically, first derivative is equal to zero what we can described in Figure 2 .

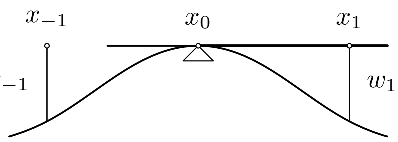


Figure 2: Substitution for unknown value at clamped edge

For free edge, we know that moment and transverse force at free edge are equal to zero. For moment, after using the second derivarive, we have

$$\begin{aligned} M_n &= -EI\frac{1}{h^2}\left(w_{n-1} - 2w_n + w_{n+1}\right) = 0 \\ &\Rightarrow \quad w_{n+1} = 2w_n - w_{n-1} \quad . \end{aligned} \quad (12)$$

The expression for transverse force is

$$T(x) = -EIw'''(x)\ . \tag{13}$$

what introduced on free edge leads to

$$\begin{aligned} T_n &= -\frac{EI}{2h^3}\left(w_{n-2} - 2w_{n-1} + 2w_{n+1} - w_{n-2}\right) = 0 \\ &\Rightarrow\ w_{n+2} = w_{n-2} - 2w_{n-1} + 2w_{n-1} \\ &\Rightarrow\ w_{n+2} = w_{n-2} - 4w_{n-1} + 4w_n\ . \end{aligned} \tag{14}$$

We expressed unknown function value for the point outside the beam with points from domain.

# 5 How to involve concentrated load in FDM scheme

The beam under concentrated force at any point doesn't have third derivative in that point. The value of transverse force close to point on the left is different to value close to the point from right. The difference is equal to given concentrated force at the point. It follows, there is also no fourth derivative at that point. The question is how to involve concentrated load in FDM scheme to get valuable solution that converges with increasing number of points. The problem has to be viewed in its physical meaning. We have to ask what we really have in that point. We have discontinouity in third derivative. That means, we do not have same value for third derivative very close left to point and very close right to point. The third derivative is not defined at point of concentrated load. The idea is to apply that difference between values for third derivative. We can write that difference as equation, the difference between transverse forces at the point of concentrated force is equal to given concentrated force

$$\Delta T_i = \left[-EIw'''\left(x\right)^{-}\right] - \left[-EIw'''\left(x\right)^{+}\right] = K\ . \tag{15}$$

Because of discontinouity at the loading point, we can not use standard scheme for third derivative. We have to introduce the expressions for third derivative only with points form the same side of point with concentrated force. That leads to following schemes for the third derivative by using the points before the point of interest,

$$w'''\left(x\right)^{-} = \frac{w_i - 3w_{i-1} + 3w_{i-2} - w_{i-3}}{h^3}\ , \tag{16}$$

and for the third derivative by using the points after the point of interest,

$$w'''\left(x\right)^{+} = \frac{-w_i + 3w_{i+1} - 3w_{i+2} + w_{i+3}}{h^3}\ . \tag{17}$$

After applying this expressions to equation (15), follows the expression for difference in transverse force in form of finite difference scheme as

$$\begin{aligned} &w_{i-3} - 3w_{i-2} + 3w_{i+1} - 2w_i \\ &\quad +3w_{i-1} - 3w_{i-2} + w_{i-3} = \frac{Kh^3}{EI}\ . \end{aligned} \tag{18}$$

# 6 Numerical example

Let we apply given procedure to simply supported beam of length $L$ and flexural rigidity $EI$ under concentrated force at midpoint $x_K = L/2$, Figure 3. We

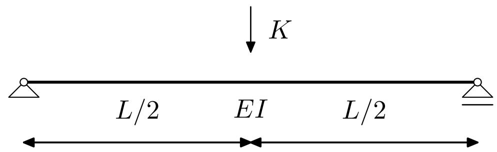


Figure 3: Simply supported beam, concentrated force at midpoint

are going to show the procedure with discretization of the beam on 9 points, $x_i = \frac{iL}{8}, i = 0, \ldots, 8$. After applying the finite difference scheme at any point, equation (8), except at the midpoint, boundary conditions ($w_0 = w_8 = 0$) and discontinuity of transverse force, equation (18), we get the system of equation,

$$\mathbf{K}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{bmatrix} = \frac{KL^3}{512EI}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\ , \tag{19}$$

with stiffness matrix calculated from FDM scheme,

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -3 & 3 & -2 & 3 & -3 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 1 & -4 & 5 \end{bmatrix}\ . \tag{20}$$

The middle row of stiffness matrix is expressed by using equation (18) for discontinouty in third derivative, in transverse force. All other row are same as in standard procesure for distributed load. After solving the system of equation, we get displacement at midpoint, $w\left(\frac{L}{2}\right) = \frac{11KL^3}{512EI}$. Analytical solution is

$w\left(\frac{L}{2}\right) = \frac{KL^3}{48EI}$. With discretization over more points, we get sequence of solution that converges to analytical solution, Table 1 .

Table 1: Displacement at mid-point

| $m$ | $w_{\frac{L}{2}} / \left(\frac{KL^3}{EI}\right)$ | error % |
|---|---|---|
| 9 | $\frac{11}{512} = \frac{1.03125}{48}$ | $3.125\%$ |
| 17 | $\frac{43}{2048} = \frac{1.00781}{48}$ | $0.78125\%$ |
| 33 | $\frac{191}{8192} = \frac{1.00195}{48}$ | $0.1953125\%$ |
| exact | $\frac{1}{48}$ | |

It is obvious that we get quadratic order of convergence.It means that double more points (double less mesh size) leads to quadruple better numerical solutions. The error is four times less than the error with former number of points with double larger mesh size.

## 7 Conclusion

The procedure for involving the concentrated force in FDM scheme for beam beding problem is introduced. The concentrated force is not applied on the beam as loading. It has been used as discontinouity in third derivative (transverse force) of the displacement function. The FDM scheme for third derivative with points from same side of loaded point is given. The difference between transverse force is given as equation that takes concentrated force into calculation. The proposed procedure has been veryfied on numerical example with different number of points to show how the sequence of numerical solution with different number of the points converges to analytical solution. The expected quadratic convergence is shown for given procedure.

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