Natural vibrations of a nano-beam with cracks

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Abstract: - Natural vibrations of nano-beams are studied. The beams under consideration have constant dimensions and are weakened by cracks or crack-like defects. Making use of a non-local theory of elasticity the influence of the crack on the vibration of the nano-beam is modelled with the aid of a massless rotating spring connecting the adjacent segments of the nano-beam.

Key-Words: - nano-beam, crack, non-local elasticity, vibration

1 Introduction

Although the idea of a non-local constitutive law in elasticity was suggested already by Eringen [5] more than thirty years ago the intensive application of non-local theories started recently with the investigations of nano-materials (see Peddieson et al [10], Reddy [11]). Reddy [11] developed non-local theories for bending, buckling and vibration of classical and nano-beams. Challamel [2]; Challamel and Wang [3] employed strain gradient elasticity and non-local elasticity models with variational approach in order to derive non-local higher-order shear theories of elastic beams. The variational formulation of these higher order models leads to the consistent system of equations and boundary conditions [2]. The non-local elastic Timoshenko-type beam theory was used in the buckling analysis of micro and nano-rods by Wang et al [14].

In the present paper a non-local elasticity type approach is developed for the investigation of nano-beams with cracks. The crack is modelled as a massless rotating spring connecting the adjacent segments of the nano-beam. The rigidity of the spring is reciprocal to the compliance of the nano-beam at the present cross section.

2 Formulation of the problem and non-local elasticity

A nano-beam of length l subjected to the impulsive loading is considered. It is assumed that the nano-beam has piece wise constant thickness $h = h_0$ for $x \in (0, a)$ and $h = h_1$ for $x \in (a, l)$. At $x = a$ a crack-like defect is located. In this study we will not pay any attention to the reasons why the crack has cropped up; we will concentrate on the determination of the influence of the crack on the vibrational characteristic of the nano-beam.

It is assumed that the material of the nano-beam is an non-linear elastic material obeying a non-local theory of elasticity. According to the concepts of Eringen [5] the constitutive equations for materials obeying the non-local elasticity can be expressed as

$$\sigma_{ij}^{n}(x) = \iiint\limits_{(V)} \alpha(|x-y|, \tau)\sigma_{ij}^{c}(y)dV \qquad (1)$$

In (1) σ_{ij}^{n} stands for the stress tensor in the non-local elasticity, σ_{ij}^{c} is the classical stress tensor and V stands for the volume of the body. Here α is the kernel function which admits to express the influence of the strain state at $y \in V$ to the stress-strain state at $x \in V$ and τ is a physical constant.
Evidently, different forms of the kernel function $\alpha(|x|)$ in (1) define different approximate models of non-local elasticity. Let us assume that $\alpha(|x|)$ is the Green's function of the linear differential operator $\mathcal{L}$. In this case

$$\mathcal{L}\alpha(|y-x|) = \delta(|y-x|) \qquad (2)$$

where δ is the Dirac's δ–function.
It was shown by Eringen [5], also Lu, Lee, Lu [9] that a simple two-dimensional kernel function can be obtained by taking

$$\mathcal{L}(\alpha) = (1 - (e_0 a)^2 \nabla^2)\alpha(x) \qquad (3)$$

where ∇ is the Laplace operator.
In (3) e_0 is a physical constant and a strands for a lattice dimension so that $e_0 a < 2$ nm.
It is reasonable to introduce the notation

$$\eta = (e_0 a)^2 \tag{4}$$

According to (1) – (4) the constitutive equation of the non-local elasticity can be expressed as

$$(1 - \eta \nabla^2)\sigma_{ij}^n = \sigma_{ij}^c \tag{5}$$

In the generalised stresses (5) has the form

$$M - \eta \frac{\partial^2 M}{\partial x^2} = M_C \tag{6}$$

where M is the bending moment in the non-local elasticity and M_C – the classical bending moment.

3 Basic equations of the vibration

According to the Kirchhoff hypotheses and the Euler-Bernoulli bending theory

$$M_C = -EI \frac{\partial^2 w}{\partial x^2} \tag{7}$$

where E is the Young's modulus and I stands for the moment of inertia of the cross section of the beam. Evidently, in the present case $I = bh^3/12$, b being the width of the nano-beam.
The equilibrium equations of a beam element can be presented as (see Soedel [12])

$$\frac{\partial M}{\partial x} = Q$$
$$\frac{\partial Q}{\partial x} = \rho bh \frac{\partial^2 w}{\partial t^2} - N \frac{\partial^2 w}{\partial x^2} \tag{8}$$

In (8) Q stands for the shear force and w is the transverse defection; ρ is the density of the material and N – the axial force.
Combining (6) – (8) leads to the equation

$$M = \eta(-Nw'' + \mu \ddot{w}) - EIw'' \tag{9}$$

where prims denote the differentiation with respect to x and

$$\ddot{w} = \frac{\partial^2 w}{\partial t^2} \tag{10}$$

Substituting (9) in (8) leads to the equation

$$w^{IV} - \frac{Nw''}{\eta N + EI} + \mu \frac{\ddot{w} - \eta \ddot{w}''}{\eta N + EI} = 0 \tag{11}$$

4 Solution of the equation of vibration

Let us consider first the case of a nano-beam of constant thickness. For solution of the equation (11) the method of separation of variables will be employed. As we are interested in studying of natural vibrations of nano-beams it is reasonable to separate the variables in the form

$$w = W(\mathrm{x}) \cos \omega t \tag{12}$$

where ω stands for the frequency of natural vibrations and $W(x)$ presents the shape of the fundamental wave.

Differentiating (12) one immediately obtains

$$\begin{aligned} w'' &= W''(x)\cos(\omega t), \\ w^{IV} &= W^{IV}(\mathrm{x})\cos \omega t, \\ \ddot{w} &= -\omega^2 W(\mathrm{x})\cos \omega t, \\ \ddot{w}'' &= -\omega^2 W''(\mathrm{x})\cos \omega t. \end{aligned} \tag{13}$$

Substitution of the partial derivatives (13) in (11) leads to the equation

$$W^{IV} - \frac{NW''}{\eta N + EI} + \frac{\mu \omega^2}{\eta N + EI}(W - \eta W'') = 0 \tag{14}$$

For solution of the linear fourth order differential equation let us compile the characteristic equation

$$\lambda_C^4 - \frac{\lambda_C^2}{\eta N + EI}(N - \mu \eta \omega^2) - \frac{\mu \omega^2}{\eta N + EI} = 0 \tag{15}$$

The solution of (15) can be presented as

$$\lambda_1 = \lambda, \lambda_2 = -\lambda, \lambda_{3,4} = \pm \beta \tag{16}$$

where

$$\lambda = \left(\frac{1}{2(\eta N + EI)}\left[N - \mu\eta\omega^2 + \sqrt{(N-\mu\eta\omega^2)^2 + 4(\eta N + EI)\mu\omega^2}\right]\right)^{\frac{1}{2}} \quad (17)$$

and

$$\beta = \left(\frac{1}{2(\eta N + EI)}\left[-N + \mu\eta\omega^2 + \sqrt{(N-\mu\eta\omega^2)^2 + 4(\eta N + EI)\mu\omega^2}\right]\right)^{\frac{1}{2}} \quad (18)$$

Thus the general solution of the homogeneous fourth order equation (14) can be presented for $x \in (0,a)$ as

$$W(x) = C_1 ch\lambda x + C_2 sh\lambda x + C_3 cos\beta x + C_4 sin\beta x \quad (19)$$

and for $x \in (a,l)$ as

$$W(x) = B_1 ch\lambda x + B_2 sh\lambda x + B_3 cos\beta x + B_4 sin\beta x \quad (20)$$

In (19), (20) $C_1 - C_4$ and $B_1 - B_4$ stand for arbitrary constants. The constants must be determined according to the continuity conditions of W, M, Q also the jump condition

$$cos\omega t\left(W'(a+) - W'(a-)\right) = KM(a) \quad (21)$$

and

$$W(a+) - W(a-) = 0,$$
$$M(a+) - M(a-) = 0, \quad (22)$$
$$Q(a+) - Q(a-) = 0.$$

In (21) the coefficient K is considered as the additional compliance caused by the crack of length $c = sh$. According to the concept developed by Chondros et al [4], also by Lellep, Kägo [6], Lellep, Kraav [7], Lellep and Liyvapuu [8] one can define K as

$$K = \frac{6\pi(1-\nu^2)}{EI} f(s). \quad (23)$$

In (23) the following notation ise used

$$f(s) = \int_0^s ZF^2(z)dz. \quad (24)$$

The stress correction function F in (24) is taken by Chondros et al as

$$F = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4. \quad (25)$$

Alternative forms of the function $F(s)$ can be found in the handbook by Tada et al [13].
Another stress correction function used by Anifantis, Dimarogonas [1] is

$$F = \frac{\sqrt{tan\psi}}{\psi cos\psi}\left[0.923 + 0.199(1 - sin\psi)^4\right] \quad (26)$$

where $\psi = \pi s/2$.
It is worthwhile to mention that the system on intermediate conditions (21), (22) must be complemented with the boundary conditions

$$W(0) = 0, W(l) = 0 \quad (27)$$

and

$$M(0) = 0, M(l) = 0. \quad (28)$$

5 Determination of eigenfrequencies

For determination of the eigenfrequencies of natural vibrations of the nano-beam one has to define the constants $C_1 - C_4$ and $B_1 - B_4$ from the requirements (21), (22) and (27). It follows from (9), (12), (13) that

$$M = -cos\omega t[(N + EI)W'' + \mu\omega^2 W]. \quad (29)$$

Combining (29) and (28)) results in

$$W''(0) = 0, W''(l) = 0. \quad (30)$$

Making use of (19), (20) and (27), (30) one easily obtains $C_1 = C_3 = 0$ and

$$B_1 = -tanh\lambda l \cdot B_2,$$
$$B_3 = -tanh\beta l \cdot B_4. \quad (31)$$

Taking into account the relations (19), (20), (29) one can rewrite the requirements (21), (22) for a stepped nano-beam as

$$\begin{gathered} W'(a+) - W'(a-) = \\ -K[W''(a+)(N + EI_1) + \mu\omega^2 W(a)], \\ W(a+) - W(a-) = 0, \\ (\eta N + EI_1)W''(a+) - \\ (\eta N + EI_0)W''(a-) = 0, \\ (N + EI_1)W'''(a+) + \mu\omega^2 W'(a+) - \\ (N + EI_0)W'''(a-) - \mu\omega^2 W'(a-) = 0. \end{gathered} \tag{32}$$

Here the following notation is used:

$$I_1 = \frac{bh_1^3}{12}, I_0 = \frac{bh_0^3}{12}. \tag{33}$$

6 Numerical results and discussion

The set of equations (31), (32) presents a linear algebraic system for determination of unknowns $C_1 - C_4$, $B_1 - B_4$. Since it is a homogeneous system its determinant must vanish. Equalizing $\nabla = 0$ one can solve the obtained equations with respect to the eigenfrequency ω.
This equation is solved numerically with the aid of the computer code MathCad.

Calculations are carried out for the nano-beam with dimensions $l = 100nm$, $h = 5nm$, $b = 1nm$. The axial tension $N = 1nN$. The material parameters are: $E = 200GPa$, $\mu = 7850\, kg/m^3$, $e_0 a = 2nm$. The results of calculations are presented in Fig.1 – 5.

Fig.1 presents the eigenfrequency ω in GHz versus the defect location a/l. Different curves in Fig.1 correspond to nano-beams with thicknesses $h = 4.5nm$, $h = 5nm$ and $h = 5.5nm$. It can be seen from Fig.1 that all the curves presented are symmetrical with respect to the central position $a = l/2$, as it might be expected.

The natural frequency ω in GHz versus the crack length c/h is depicted in Fig.2 for different positions of the crack. It can be seen from Fig.2 that the natural frequency ω increases when the crack position moves towards an edge of the nano-beam. It reveals from Fig.2 that the crack extension (length) only slightly influences on the fundamental frequency if the crack is located near the edge of the nano-beam.

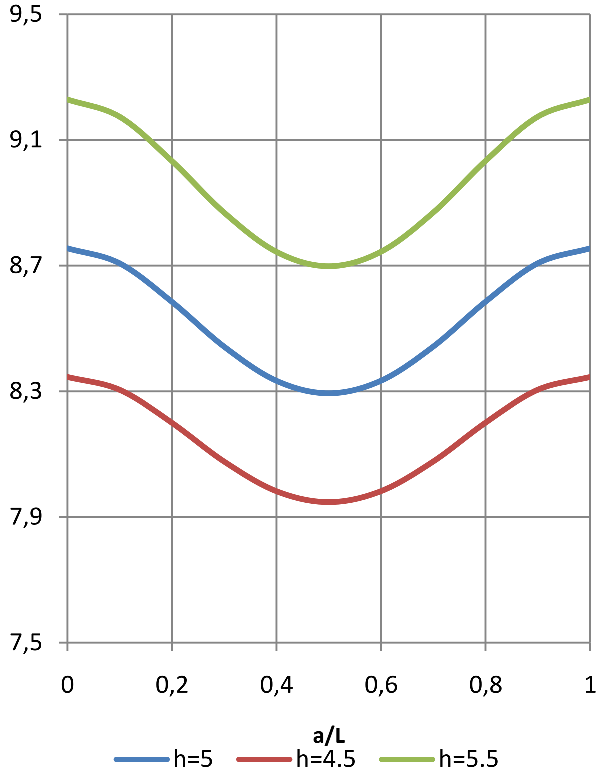

Fig.1. Eigenfrequency $\omega\,(GHz)$ versus the defect location a/L at different beam thicknesses.

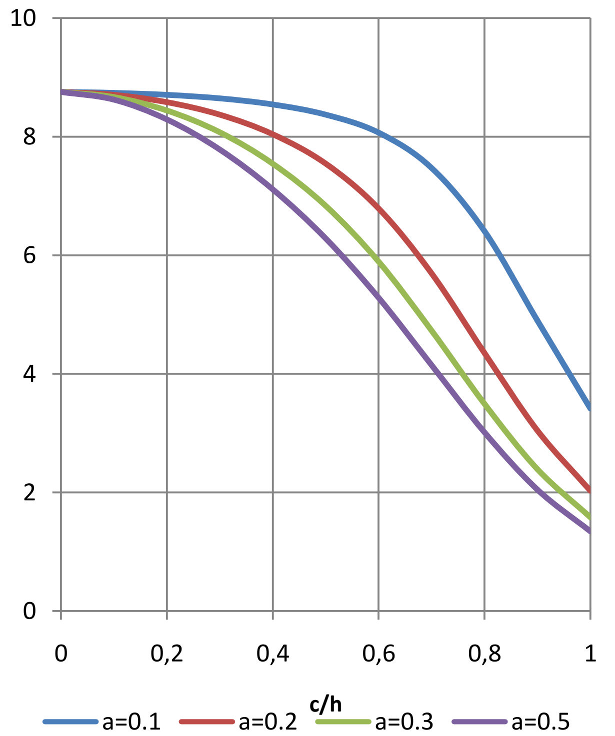

Fig.2. Influence of the crack extension on the eigenfrequency $\omega\,(GHz)$ depending on the crack location.

The influence of the parameter $e_0 a$ on the frequency of the natural vibrations ω is depicted in Fig.3. The result are shown as a ratio ω/ω_0, where ω_0 is the natural frequency at $e_0 a = 0$ and a given crack location. It shows that the eigenfrequency slightly depends on the position of the crack in the nano-beam.

Fig.4 depicts the influence of the thickness of the beam on the frequency of the natural vibrations ω. A ratio ω/ω_1 is used depict the results, where ω_1 is the natural frequency at $h = 4.5nm$ and a given crack location. Fig. 4 shows that the natural frequency strongly depends on the thickness of the nano-beam, as it might be expected. Different curves in Fig.3and Fig.4 correspond to the location of the crack at $x = 0$, $x = 0.25l$ and $x = 0.5l$.

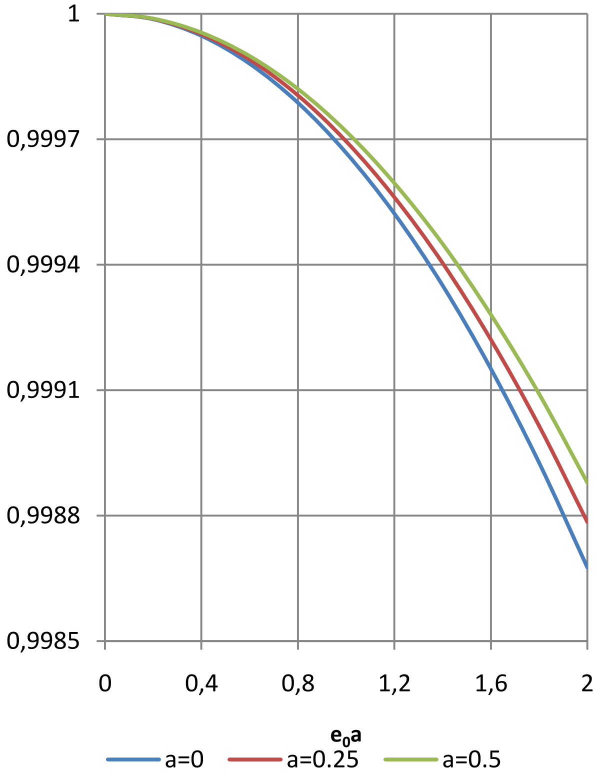

Fig.3. Ratio ω/ω_0 depending on the parameter $e_0 a$ and position of the crack.

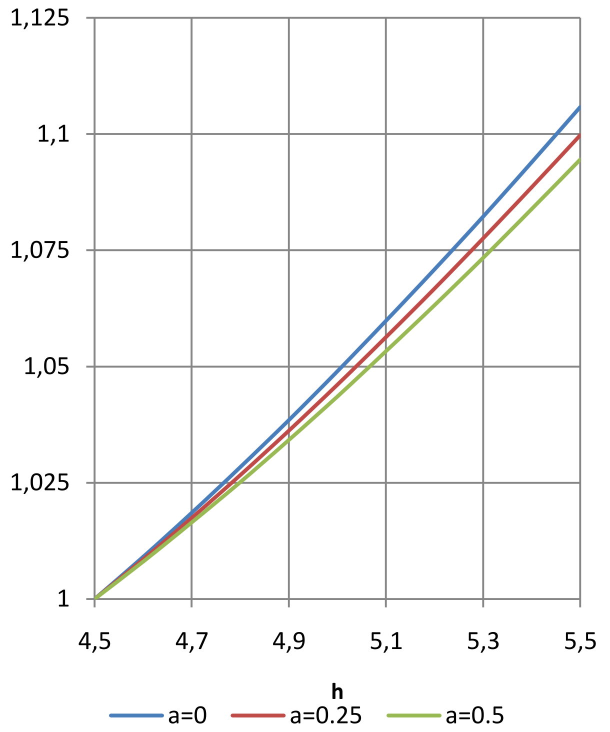

Fig.4. Ratio ω/ω_1 depending on the thickness of the beam at given crack locations.

In Fig.5 the influence of the length of the nano-beam on the eigenfrequency ω (GHz) is depicted. One can see from Fig.5 that the shorter is the nano-beam the higher is the eigenfrequency.

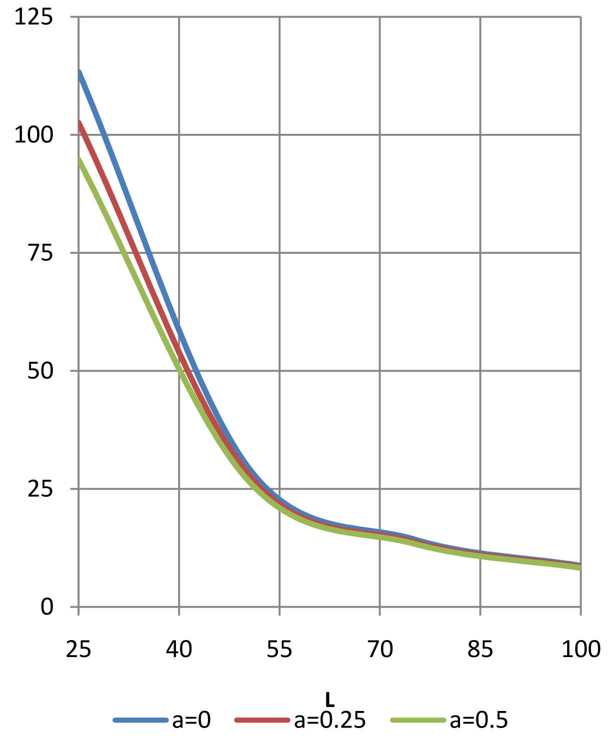

Fig.5. Eigenfrequency ω (GHz) versus the length of the beam L depending on the crack location.

7 Concluding remarks

A method for determination of natural frequencies of nano-beams is developed. The nano-beams under consideration have constant dimensions and are weakened by stable cracks.

It is shown that the cracks affect the frequencies of natural vibrations of nano-beams.

The influence of a crack on the vibration of nano-beam is less remarkable if the crack is located near a simply supported edge of the nano-beam. Calculations carried out showed that the eigenfrequency monotonically decreases if the length of the nano-beam increases.

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