A Study on Smooth Varieties with Differentially Simple Coordinate Rings

MICHAEl GR. VOSKOGLOU
Department of Mathematical Science3s
Graduate Technological Educational Institute of Western Greece
Meg. Alexandrou 1 - 263 34 Patras
GREECE
mvosk@hol.gr ; http://eclass.teipat.gr/eclass/courses/523102http://eclass.teipat.gr/eclass/courses/523102

Abstract: - The notion of the algebraic sets was proved to be fundamental for the development of the Algebraic Geometry. In the paper at hands we study irreducible algebraic sets (varieties) with differentially simple coordinate rings. Starting from the fact that the coordinate ring of a singular variety does not admit simple derivations, we turn our attention to smooth varieties proving that the coordinate rings of the circle of the cylinder and of the real torus are differentially simple rings. However, this is not true for the coordinate rings of all smooth varieties; for example the coordinate ring of the real sphere does not admit simple derivations.

Keywords: **-** Derivations, Differentially simple rings, Algebraic sets, Singularities, Coordinate rings of smooth varieties.

1. Introduction

All he rings considered in this paper are commutative with identity and all the fields are of characteristic zero, while a ***local ring*** is understood to be a Noetherian ring with a unique maximal ideal. For general facts on commutative ring theory we refer to the books [1-3].

A prime ideal P of a ring R is said to be of ***height n***, where n is an integer, $n\geq 1$, if there exists a chain of distinct prime ideals decreasing from P of the form $P\supset P_1\supset P_2\supset \dots \supset P_n$ and no longer one. We write then ht P = n

Further, the ***Krull dimension of R,*** denoted by dim R, is defined to be the supremum of heights of all maximal ideals of R. Note that, if R is a ***finitely generated algebra*** over a field K, then all its maximal ideals have the same height. Therefore, if M is a maximal ideal of R, then dim R = ht M.

A local ring is said to be ***regular*** if its dimension is equal to the minimal number of generators, say V(M), of its unique maximal ideal M. Generalizing, a ring R is said to be regular, if all its localizations R_M with respect to a maximal ideal M are regular local rings.

In the paper at hands we study smooth varieties over a field K, whose coordinate rings admit simple K-derivations. The rest of the paper is organized as follows: Section II contains a brief account about the differential simplicity of a ring, which is needed for the purposes of the present work. In Section III some ideas from Algebraic Geometry are exposed, which are useful for the better understanding of the paper. Finally, Section IV contains the main paper's results concerning the differential simplicity of the coordinate rings of a circle, of a cylinder and of the real torus.

2. Differentially Simple Rings

Let R be a ring and let d be a derivation of R. Then an ideal I of R is called a ***d-ideal*** if d (I)$\subseteq$I, and R is called a ***d-simple ring*** if it has no non zero, proper d-ideals. In the last case we say that d is a ***simple derivation*** of R.

For reasons of brevity we shall write dI instead of d (I).

A d-simple ring R contains the field F = {x$\in$R: dx = 0} and therefore it is either of characteristic zero, or of a prime number p.

In earlier works [4, 5] we have studied the d-simplicity of a commutative ring. If R is a d-simple ring of characteristic p, then things are quite simple; namely R is a 0-dimensional quasi-local ring and therefore, if R is a domain, then R is a field [5; Theorem 1.5].

On the contrary, if R is of characteristic zero and d is a derivation of R, then no general criterion is known to decide whether or not R is a d-simple ring, unless if R is a one-dimensional (as a ring) finitely generated algebra of the form $K[y_1, y_2,\ldots, y_n]$ over a field K . In fact, in this case R is a d-simple ring, if, and only if, $R = (dy_1, dy_2,\ldots, dy_n)$ [5; Theorem 2.4].

Several examples of d-simple rings of dimension greater than one and even of infinite dimension are exposed in [4, 5], such as the polynomial rings in finitely and infinitely many variables and the Laurent polynomial rings over a field K, the regular local rings of finitely generated type [6] , etc.

Another important result that we are going to use later in this paper states that, if R is a d-simple ***G-ring*** - a wide class of rings containing all finitely generated algebras over fields and all complete local rings, and being closed under localization [3; pp. 249-257] - of characteristic zero, then R is a regular ring [7; Theorem 1].

3. Ideas from Algebraic Geometry

This section contains the background from Algebraic Geometry which is necessary for the understanding of the rest of the paper. For general facts on Algebraic Geometry we refer to the books [8, 9].

Definition 1: Let K be a field. Then the Cartesian product

$K^n = k \times k \times\ldots\times k$ (n-times)

is called an ***affine space*** over K, and its elements

$a = (a_1, a_2,\ldots, a_n)$,

are called ***points*** of K^n, with $a_1, a_2,\ldots, a_n$ in K being the ***coordinates*** of the point a. .

Definition 2: Let $K[x_1, x_2,\ldots,x_n]$ be a polynomial ring over a field K . Then a subset Y of K^n is called an ***algebraic set*** over K^n**,** if there exists a non-empty subset S of $K[x_1, x_2,\ldots,x_n]$ such that

$Y = \{a\in K^n: f(a) = 0, \forall f \in S\}$.

We write then Y = U(S), while, if S = {f} is a singleton set, we write for simplicity Y = U (f).

Examples: $U(0) = K^n$, $U(1) = \varnothing$, $U(x_1^2+x_2^2-1)$ = unit circle, $U(x_2-x_1^2)$ = parabola, $U(x_1^2+x_2^2+x_3^2-1)$ = unit sphere, $U(x_1^2-x_2^2-x_3^2)$ = cone, etc.

Definition 3: Let Y be an algebraic set of an affine space over a field K. Then, it is easy to check that the set

$I = J(Y) = \{f\in K[x_1, x_2,\ldots,x_n]: f(a)=0, \forall a\in Y\}$

is an ideal of $K[x_1, x_2,\ldots,x_n]$, called the ***ideal of Y.***

As a special case $J(\varnothing) = K[x_1, x_2,\ldots,x_n]$, while if K is an infinite field , then $J(K^n) = 0$.

Obviously, if Y= U (S), then $J(Y)\supseteq S$. Further we have:

Proposition 4: If Y is an algebraic set over K^n, then

Y = U(J(Y))

Proof: If $a\in Y$, then, by Definition 3, f(a) = 0, for all f in J(Y). Hence, by definition 2, a is in U(I(Y)), therefore $Y \subseteq U(J(Y))$.

Conversely, let Y = U(S) and let $a\in U(J(Y))$. Then, by Definition 2, f(a) = 0 for all f in J(Y). But $J(Y)\supseteq S$, therefore f(a) = 0 for all f in S, which means that $a\in Y$. Thus $U(J(Y)) \subseteq Y$.

Definition 5: Let Y be an algebraic set of an affine space over a field K and let I= J(Y). Then the factor ring

$R = K[x_1, x_2,\ldots,x_n] / I$

is called the ***coordinate ring of Y.*** Further, the ***dimension of Y*** is defined to be equal to the dimension of R.

Definition 6: Let Y be an algebraic set of an affine space over a field K, let R be the coordinate ring of Y and let a be a point of Y. It is easy then to check that the set

$J(a) = \{f\in K[x_1, x_2,\ldots,x_n]: f(a) = 0\}$

is a maximal ideal of the polynomial ring $K[x_1, x_2,\ldots,x_n]$. Therefore, the image $M_a = J(a)/I$ of J(a) in R is a maximal ideal of R. Then, the local ring $R_a = R_{M_a}$ is called ***the local ring of a in Y.***

Definition 7: An algebraic set of an affine space over a field K is said to be ***irreducible*** if it cannot be written as the union of two smaller algebraic sets.

In this paper an irreducible algebraic set is called a ***variety.***

Definition 8: Let Y be a variety over K^n and let a be a point of Y. Then an element $(b_1, b_2, \ldots, b_n)$ of K^n is said to be a ***tangent to Y at a***, if for all f in J(Y) we have that

$$\sum_j \frac{\partial f(a)}{\partial x_j} = 0.$$

It is easy to check that the set Ta of all tangents of Y at a is a vector space over K called the ***tangent space of Y at a.***.

It is well known that the dimension of the vector space T_a is equal to the minimal number, say $V(M_a)$, of generators of the maximal ideal M_a of the local ring of a in Y (Definition 6). But, by the generalized principal ideal theorem [2; Theorem 152) ht $M_a \leq V(M_a)$. But R is a finitely generated algebra, therefore dim R = ht M_a, therefore

dim R $\leq$ dim Ta.

Definition 9: Let Y be a variety over K^n. Then a point a of Y is said to be a ***simple point***, if the dimension of Y is equal to the dimension of the tangent space T_a. Otherwise a is called a ***singular point*** of Y.

A variety Y has always simple points, while it may have or not singular points.

Definition 10: A variety Y over K^n not having singular points is called a ***smooth variety***, otherwise it is called a ***singular variety.***

It is now easy to prove the following result:

Proposition 11: If the coordinate ring of a variety Y over K^n is a regular ring, then Y is a smooth variety.

Proof: Let a be any point of Y and let R_a be the local ring of a in Y. Since R is a regular ring, dim $R_a = V(M_a)$. But R is a finitely generated K-algebra, therefore

dim R = ht M_a .= dim $R_a = V(M_a)$ = dim T_a.

This shows that a is a simple point of Y, therefore Y is a smooth variety.

4. Main results

In this section we present examples of smooth varieties, whose coordinate rings admit simple derivations, i.e. they are differentially simple rings. We emphasize that this is not always true; e.g. it is well known that the coordinate ring

$$S=\frac{IR[x_1, x_2, x_3]}{(x_1^2+x_2^2+x_3^2-1)}$$

of the ***real sphere***, although it is regular, admits no derivation d , such that S is d-simple. [7; Section 3, Example (iii)].

We start with the following result showing that only the coordinate rings of smooth varieties can admit simple derivations:

Theorem 12: Let k be a field, let n be a non negative integer, and let Y be a singular variety over k^n . Then the coordinate ring R of Y admits no simple derivations.

Proof: Assume that there exists a derivation d of R, such that R is a d-simple ring. But R, being a finitely generated K-algebra is a G-ring, therefore R is a regular ring [7; Theorem 1]. Consequently, by Proposition 11, Y is a smooth variety, which contradicts our hypothesis..

Next we are going to present some characteristic examples of smooth varieties having d-simple coordinate rings, for suitably chosen k-derivations d of them.

First, we study the case of the ***circle***:

Theorem 13: The coordinate ring

$$R=\frac{k[x_1, x_2]}{(x_1^2+x_2^2-1)}$$

of the unit circle defined over a field k admits k-derivations d such that R is a d-simple ring.

Proof: Since P = $(x_1^2+x_2^2-1)$ is a prime ideal of k[x_1, x_2], we have that dim R = 2 – ht P and therefore, by the principal ideal theorem [2; Theorem 142), dim R = 1.

Consider the k-derivation d of k[x_1, x_2] defined by

$$d\,x_1 = a\,x_2,\; d\,x_2 = -a\,x_1,$$

for some non zero element a of k. Since dP $\subseteq$ P, d induces a k-derivation of R, denoted also by d.

Given f in k[x_1, x_2], set $\overline{f}$ = f + P, then

$$\overline{x_2}\, d\overline{x_1} - \overline{x_1}\, d\overline{x_2} = a.$$

Thus the result follows by Theorem 2.4 of [5], stating that, if d is a k-derivation of an one-dimensional finitely generated k-algebra, say R=k[$y_1, y_2, ..., y_n$], then R is d-simple, if, and only if, R = ($dy_1, dy_2, ..., dy_n$).

Next, we consider the case of the ***cylinder***:

Theorem 14: The coordinate ring

$$C = \frac{k[x_1, x_2, x_3]}{(x_1^2 + x_2^2 - 1)}$$

of the cylinder defined over a field k is d-simple for suitable k-derivation d of C.

Proof: Reconsider the coordinate ring R of the unit circle. We can write

$$R = k\,[\,\overline{x_1}, \overline{x_2}\,]$$

and

$$C = k\,[\,\overline{x_1}, \overline{x_2}, x_3\,] = R\,[x_3].$$

The derivation

$$d = \frac{\partial}{\partial x_1} x_2 - \frac{\partial}{\partial x_2} x_1$$

of k [x_1, x_2] induces a derivation of R, denoted also by d.

The above derivation d is the special case of the derivation of Theorem 13 for a = 1, therefore R is a d-simple ring.

We shall show that d can be extended to a derivation of C, such that C is also a d-simple ring.

For this, observe first that any F in R can be written in the form

$$F = \sum_{i=0}^{n} \overline{g_i}\; \overline{x_2}^{\,i},$$

where n is a non negative integer and g_i is in k [x_1], for each i.

But

$$\overline{x_2}^{\,2} = 1 - \overline{x_1}^{\,2},$$

therefore

$$F = \overline{f}\;\overline{x_2} + \overline{g},$$

with f, g in k [x_1].

This expression of f is unique, because

$$\overline{f}\;\overline{x_2} + \overline{g} = \overline{0}$$

gives that

$$f\,x_2 + g = h(x_1^2 + x_2^2 - 1).$$

Thus, on comparing the degrees of both sides with respect to x_2, it turns out that h = 0 and therefore f = g = 0.

We shall show further that dF $\neq \overline{1}$, for all F in R. In fact,

$$dF = d\,\overline{f x_2} + \overline{f}\, d\overline{x_2} + d\,\overline{g}$$

$$= \frac{\partial \overline{f}}{\partial \overline{x_1}} \overline{x_2}^{\,2} - \overline{f x_1} + \frac{\partial \overline{g}}{\partial \overline{x_1}} \overline{x_2} = \frac{\partial \overline{f}}{\partial \overline{x_1}} (1 - \overline{x_1}^{\,2}) - \overline{f x_1} + \frac{\partial \overline{g}}{\partial \overline{x_1}} \overline{x_2}$$

Thus, if dF= $\overline{1}$, by the uniqueness of this expression, it follows that

$$\frac{\partial \overline{g}}{\partial \overline{x_1}} = \overline{0}$$

and

$$(1-\overline{x_1}^2)\frac{\partial \overline{f}}{\partial \overline{x_1}} - \overline{f}\ \overline{x_1} = \bar{1} \qquad \textbf{(1)}.$$

But

$$f = \sum_{i=0}^{t} a_i x_1^{\ i}$$

for some non negative integer t, with α_i in k, for each i. Thus, equating the leading coefficients in (1), we find that

$$-(t+1)\alpha_t = 0.$$

Therefore $\alpha_t = 0$, which contradicts (1).

Consider now the derivation

$$\frac{\partial}{\partial x_1}x_2 - \frac{\partial}{\partial x_2}x_1 + \frac{\partial}{\partial x_3}$$

of k $[x_1, x_2.x_3]$, which induces a derivation of C, whose restriction in R is d. Denote the above derivation by d as well. We shall show then that C is a d-simple ring.

In fact, assume that I is a proper non zero d-ideal of C = R[x_3] and let A be the set of the leading coefficients of all the polynomials of least degree, say n, in I.. Obviously then A is a proper non zero ideal of R

Let a be in A, then there is a polynomial F in I, such that F = a$x_3^{\ n}$ + terms of lower degree.

But dF = (da) $x_3^{\ n}$ + terms of lower degree is also in I and therefore da is in A.Hence A is a d-ideal of R, so A=R.

This means that there exists a monic polynomial, say G=$x_3^{\ n} + a_{n-1}x_3^{\ n-1}$ + terms of lower degree, in I.

Then dG =n$x_3^{\ n-1} + d(a_{n-1})x_3^{\ n-1}$ + terms of lower degree is also in I and has degree less than n, therefore d G = 0. In particular

$$n + d(a_{n-1}) = \overline{0},$$

or

$$d\left(-\frac{1}{n}a_{n-1}\right) = \bar{1},$$

a contradiction, since

$$-\frac{1}{n}a_{n-1} = \text{F is in R}.$$

Remark: The varieties whose ideals are principal, like the circle and the cylinder, are called ***hypersurfaces.***

Next we study the case of the ***real torus*** obtained by rotating a cycle around an axis in its plane which does not intersect it (Figure 1).

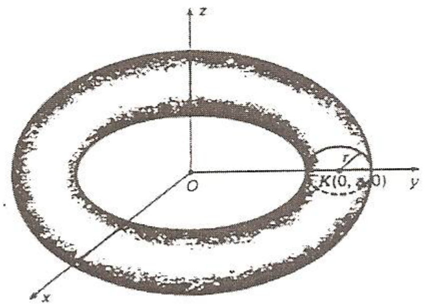

Figure 1: The surface of a torus

For this, we need the following Lemma:

Lemma 15: Let A= ***R*** $[x_1, x_2, ..., x_{2n}]$ be a polynomial ring over the field ***R*** of the real numbers. Then the ***R*** -derivation d of A defined by

$$d\ x_{2i-1} = a_i x_{2i},\ d(x_{2i}) = -a_i x_{2i-1},$$

with a_i in ***R*** for each i = 1, 2,…n, induces a derivation of the ring

$$T = \frac{A}{(x_1^{\ 2} + x_2^{\ 2} - 1, ..., x_{2n-1}^{\ \ 2} + x_{2n}^{\ 2} - 1)},$$

denoted also by d, such that T is d-simple if, and only if, $a_1, a_2, ..., a_n$ are linearly independent over the ring ***Z*** of integers.

Proof: Set

$$\mathrm{I}=(x_1^{\ 2}+x_2^{\ 2}-1,\ldots..x_{2n-1}^{\ 2}+x_{2n}^{\ 2}-1).$$

Since

$$\mathrm{d}\,(\mathrm{x}_{2i-1}^{\ 2}+x_{2i}^{\ 2}-1)=0$$

for each i, is d $\mathrm{I}\subseteq I$, and therefore d induces a derivation of $\mathrm{T}=\dfrac{A}{I}$.

Consider now the polynomial ring

B=$\boldsymbol{C}\,[x_1, x_2,\ldots,x_{2n}]$, where $\boldsymbol{C}$ denotes the field of complex numbers and set

I^e = BI. Then d extends to a $\boldsymbol{C}$-derivation of B by

$$\mathrm{d}\,(\mathrm{f}+\mathrm{ig})=\mathrm{df}+\mathrm{idg}$$

for all f, g in A. It is easy to check that I^e is a d-ideal of B, therefore d lifts to a $\boldsymbol{C}$-derivation of $\dfrac{B}{I^e}$.

It is straightforward to show that $\dfrac{A}{I}$ is a d-simple ring, if, and only f, $\dfrac{B}{I^e}$ is a d-simple ring.

Set

$$\mathrm{y}_j=(x_{2j-1}+ix_{2j})+I^e$$

and

$$\mathrm{y}_j^{-1}=(x_{2j-1}-ix_{2j})+I^e,$$

for j = 1,2,….,n. Then we have that

$$x_{2j-1}+I^e=\frac{y_j+y_j^{-1}}{2},\quad x_{2j}+I^e=\frac{y_j-y_j^{-1}}{2},$$

and

$$\mathrm{y}_j y_j^{-1}=1+I^e,$$

for each j and therefore we can write

$$\frac{B}{I^e}=\mathbf{C}\,[\mathrm{y}_1,\mathrm{y}_1^{-1},\ldots..,y_n,y_n^{-1}].$$

But

$$\mathrm{d}\,(\mathrm{y}_j)=[dx_{2j-1}+id(x_{2j})]+I^e$$

$$=(a_jx_{2j}-ia_jx_{2j-1}+\mathrm{I}^e=-ia_jy_j=b_jy_j.$$

Therefore the result follows by Theorem 3.5 of [4], stating that $\boldsymbol{C}\,[\mathrm{y}_1,\mathrm{y}_1^{-1},\ldots..,y_n,y_n^{-1}]$ is d-simple, if, and only if, the b_j's are linearly independent over ***Z.***

For n =2 the ring

$$\mathrm{T}=\frac{R\,[x_1,x_2,x_3,x_4]}{(x_1^{\ 2}+x_2^{\ 2}-1,x_3^{\ 2}+x_4^{\ 2}-1)}$$

of Lemma 15 is the coordinate ring of the real torus, considered as a 2-dimensional surface in 4 dimensions. Thus one obtains the following result:

Theorem 16: The coordinate ring T of the real torus is d-simple for suitable R-derivations d of T.

Proof: Set

$$\mathrm{d}=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}+ax_4\frac{\partial}{\partial x_3}-ax_3\frac{\partial}{\partial x_4},$$

where a is an irrational number and apply Lemma.15.

Remark: Combining the above results with Corollary 3.6 of [10] one obtains examples of ***simple skew polynomial rings*** over the coordinate rings of the circle, the cylinder and the real torus.

References

[1] Atiyah, M. F. & MacDonald, I. G., *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969.
[2} Kaplansky, I., *Commutative Rings*, Revised Edition, The Univ. of Chicago Press, Chicago and London, 1974.
[3] Matsumura H., *Commutative Algebra*, Second Edition, Benjamin/Cummings, Reading, Mass.., 1980.
[4] Voskoglou M.Gr., Differential simplicity and dimension of a commutative ring, *Riv. Mat.*

Univ.. Parma,, 6(4**)**, 111-119, 2001.

[5] Voskoglou M.Gr., Derivations and Iterated Skew Polynomial Rings, *NAUN International Journal of Applied Mathematics and Informatics*, 6(2), 2011, pp. 32-90.

[6] Hart, R., Derivations on regular local rings of finitely generated type, *J. London Math. Soc.*, 10, 1975, pp. 292-294.

[7] Hart, R., Derivations on commutative rings, *J. London Math. Soc.*, 8, 1974, pp. 171-175.

[8] Fulton W., *Algebraic Curves: An introduction to Algebraic Geometry*, Benjamin / Cummings, Reading, Mass. , 1978

[9] Mumford, D.., *Algebraic Geometry I* **-** *Complex projective varieties*, Springer, New York, 1995

[10] Voskoglou M.Gr.., Simple skew polynomial rings, *Publ. I. Math. (Beograd),* 37(51), 1985, pp. 37-41.