Super-$\lambda_3$ and super-$\kappa_3$ graphs on girth and diameter

Litao Guo
School of Applied Mathematics
Xiamen University of Technology
Xiamen Fujian 361024
P.R.China
ltguo2012@126.com

Xiaofeng Guo
School of Mathematical Sciences
Xiamen University
Xiamen Fujian 361005
P.R.China
xfguo@xmu.edu.cn

Abstract: Let $G = (V, E)$ be a connected graph. An edge set $F \subseteq E$ is a 3-restricted edge cut, if $G − F$ is disconnected and every component of $G − F$ has at least three vertices. The 3-restricted edge connectivity $\lambda_3(G)$ of $G$ is the cardinality of a minimum 3-restricted edge cut of $G$. A graph $G$ is called $\lambda_3$-optimal, if $\lambda_3(G) = \xi_3(G)$, where $\xi_3(G)$ is the minimum number of edges between a connected subgraph $A$ with three vertices and $G − A$. A graph $G$ is $\lambda_3$-connected, if $G$ contains a 3-restricted edge cut. A $\lambda_3$-connected graph $G$ is said to be super-$\lambda_3$, if every minimum 3-restricted edge cut isolates a component with exactly three vertices. It is analogous to define $\kappa_3(G)$ and $\kappa_3$-connected graph $G$ for the case of vertex. A $\kappa_3$-connected graph $G$ is said to be super-$\kappa_3$, if $\kappa_3(G) = \xi_3(G)$ and the deletion of a minimum 3-restricted cut isolates a component with exactly three vertices. Let $G$ be a connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. We show that: (1) If diameter $D(G) \leq g − 4$, then $G$ is super-$\lambda_3$. (2) If diameter $D(G) \leq g − 5$, then $G$ is super-$\kappa_3$. Similar results are also obtained relating the diameter, the girth and the super connectivity of a line graph.

Key–Words: 3-Restricted edge connectivity; Super-$\lambda_3$; Super-$\kappa_3$

1 Introduction

It is well known that graph theory plays a key role in the analysis and design of reliable or invulnerable networks. A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability. Connectivity is a parameter to measure the reliability of networks.

In this paper, we only consider simple graphs. Let $G = (V, E)$ be a connected graph. For a vertex $v \in V$, $N(v)$ is the set of all vertices adjacent to $v$. The degree of a vertex $v$, denoted by $d(v)$, is the size of $N(v)$. If $u, v \in V$, then $d(u, v)$ denotes the length of a shortest $(u, v)$-path. For $X, Y \subseteq V$, $d(X, Y)$ denotes the distance between $X$ and $Y$; more formally, $d(X, Y) = \min\{d(x, y) : x \in X \text{ and } y \in Y\}$. If $v \in V$, $r \geq 0$ is an integer, then let $N_r(v) = \{w \in V : d(w, v) = r\}$, in particular, $N_1(v) = N(v)$. For $X \subseteq V$, $N_r(X) = \{w \in V : d(w, X) = r\}$, where $d(w, X) = d(\{w\}, X)$, and $N_1(X) = N(X)$. We denote the diameter and girth by $D$ and $g$, respectively, and write $G − v$ for $G − \{v\}$. A path is called $k$-path, if its length is $k$. For $U \subseteq V$, $G[U]$ is the subgraph of $G$ induced by the vertex subset $U$, and $[U, V − U]$ is the set of edges with one end in $U$ and the other in $V − U$. And $\xi_k(G) = \min\{|[U, V − U]) : U \subseteq V, |U| = k \text{ and } G[U] \text{ is connected}\}$.

Recall that for every graph $G$ we have $\lambda \leq \delta$, where $\delta$ is the minimum degree of $G$. If $\lambda = \delta$, then $G$ is said to be maximally edge connected or $\lambda$-optimal. In the definitions of $\lambda(G)$, no restrictions are imposed on the components of $G − S$, where $S$ is an edge cut. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G − S$. Following this idea, $k$-restricted edge connectivity were proposed in [3,4]. An edge set $F \subseteq E$ is said to be a $k$-restricted edge cut, if $G − F$ is disconnected and every component of $G − F$ has at least $k$ vertices. The $k$-restricted edge connectivity of $G$, denoted by $\lambda_k(G)$, is the cardinality of a minimum $k$-restricted edge cut of $G$. If $|F| = \lambda_k$, then $F$ is called a $\lambda_k$-cut. Not all connected graphs have $\lambda_k$-cuts ($k \geq 2$), for example $K_{1, n−1}$. A graph $G$ is $\lambda_k$-connected, if $G$ contains a $k$-restricted edge cut. A $\lambda_k$-connected graph $G$ is called $\lambda_k$-optimal, if $\lambda_k(G) = \xi_k(G)$.

An vertex set $X$ is a $k$-restricted cut of $G$, if $G − X$ is not connected and every component of $G − X$ has at least $k$ vertices. The $k$-restricted connectivity $\kappa_k(G)$ (in short $\kappa_k$) of $G$, is the cardinality of a minimum $k$-restricted cut of $G$. And $X$ is called a $\kappa_k$-
cut, if $|X| = \kappa_k$. Not all connected graphs have $\kappa_k$-cuts ($k \geq 2$), for example $K_{1,n-1}$. A graph $G$ is $\kappa_k$-connected, if a $\kappa_k$-cut exists. For $k = 1, 2$ we can see [1, 2, 8]. We will study the case of $k = 3$.

For $X \subset V$, $v \in V \setminus X$ and $u \in N(v)$. Let us introduce the sets $X_u^+(v) = \{z \in N(v) - u : d(z, X) = d(v, X) - 1\}$, $X_u^-(v) = \{z \in N(v) - u : d(z, X) = d(v, X)\}$, $X_u^+(v) = \{z \in N(v) - u : d(z, X) = d(v, X) + 1\}$. Clearly, $X_u^+(v), X_u^+(v)$ and $X_u^-(v)$ form a partition of $N(v) - u$. And $|X_u^+(v)| + |X_u^-(v)| = d(v) - 1$. If $d(v) \geq 2$, $u, w \in N(v)$, then $X_u^+(v) = \{z \in N(v) - \{u, w\} : d(z, X) = d(v, X) + 1\}$, $X_u^+(v) = \{z \in N(v) - \{u, w\} : d(z, X) = d(v, X)\}$. Then $X_u^+(v), X_u^+(v)$ and $X_u^+(v)$ form a partition of $N(v) - \{u, w\}$, and $|X_u^+(v)| + |X_u^+(v)| + |X_u^-(v)| = d(v) - 2$.

Wang et al.[7] obtain the following result for $\lambda_3(G)$.

**Theorem 1.1.** Let $G$ be a simple connected graph of order $n \geq 6$. If $G$ is not a subgraph of any of the graphs shown in Fig.1, then both $\lambda_3(G)$ is well defined and $\lambda_3(G) \leq \xi_3(G)$.

![Fig. 1](image)

From this theorem we can see that if $G$ is a connected graph with girth $g \geq 4$ and $\delta \geq 3$, then $G$ has $3$-restricted edge cuts.

We also have the following results for $\lambda_3(G)$ and $\kappa_3(G)$.

**Theorem 1.2.** (1) [5] Let $G$ be a $\lambda_3$-connected graph with $g \geq 4$, minimum degree $\delta \geq 3$ and diameter $D$. If $D \leq g - 3$, then $G$ is $\lambda_3$-optimal.

(2) [6] Let $G$ be a connected graph with $g \geq 6$, and minimum degree $\delta \geq 3$. Then $G$ is $\kappa_3$-connected and $\kappa_3(G) \leq \xi_3(G)$, if $g \geq 7$ or $\delta \geq 4$.

(3) [6] Let $G$ be a $\kappa_3$-connected graph with $g \geq 4$, minimum degree $\delta \geq 3$ and diameter $D$. If $D \leq g - 4$, then $\kappa_3(G) = \xi_3(G)$.

In this paper, we investigate super-$\lambda_3$ connectivity and super-$\kappa_3$ connectivity of graphs with girth $g \geq 4$ and minimum degree $\delta \geq 3$. Some sufficient conditions for the graphs to be super-$\lambda_3$ (resp. super-$\kappa_3$) are given in Theorem 3.1, which depends on diameters of the graphs and their line graphs.

In Section 2 we shall give some properties of 3-restricted edge cuts and 3-restricted cuts of graphs, in Section 3 we prove the sufficient conditions in Theorem 3.1 for graphs to be super-$\lambda_3$ (resp. super-$\kappa_3$).

**2 Properties of 3-restricted edge cuts and 3-restricted cuts of graphs**

If $G$ is a graph with girth $g \geq 4$, then every connected subgraph of $G$ with three vertices is a path $xyz$ of length two. Thus, $\xi_3(G) = \min \{d(x) + d(y) + d(z) - 4 : xzy$ is a path of length two in $G\}$.

**Lemma 2.1.** Let $G$ be a connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and $\xi_3(G)$. Let $X \subset V$ be a vertex cut with $|X| \leq \xi_3(G)$ and $C$ be any connected component of $G - X$ with $|V(C)| \geq 3$. Then the following assertions hold:

1. There exists an edge $uv$ in $C$ such that $d(\{u, v\}, X) \geq [(g - 4)/2]$.
2. If $g$ is odd and $|V(C)| \geq 4$, then there is a vertex $u \in C$ with $d(u, X) \geq (g - 5)/2$ such that $|N_{(g-5)/2}(u) \cap X| \leq 1$.

**Proof.** For $g = 4, 5, 6$, both assertions of the lemma hold, since $d(u, X) \geq 1$ for all $u \in C$ and $|V(C)| \geq 3$. So suppose that $g \geq 7$ and let $\mu = \max \{d(u, X) : u \in V(C)\}$. Note that $\mu \geq 1$. If $\mu \geq [(g - 2)/2]$, then both assertions clearly hold. Thus, we assume that $\mu \leq [(g - 4)/2]$.

(1) If $\mu = 1$, then the result holds. Thus assume that $\mu \geq 2$.

**Claim 1.** There is an edge $uv$ in $C$ such that $d(\{u, v\}, X) = \mu$.

We argue by contradiction. Suppose that each vertex $u$ in $C$ at $d(u, X) = \mu$ satisfies $d(v, X) = \mu - 1$ for all $v \in N(u)$. As $\delta \geq 3$, take $w, v \in N(u)$, then $uvw$ is a 2-path in $C$. Thus $d(v, X) = d(w, X) = \mu - 1$. Each vertex in $N(X_u^+(w))$ and $N(X_u^+(v))$ is at distance $\mu - 1$ from $X$. Moreover, we have $|N_{\mu-1}(X_u^+(w)) \cap X| \geq |X_u^+(w)|$. Otherwise, there are two vertices $x_1, x_2 \in X_u^+(w)$ both at distance $\mu - 1$ from a vertex $x \in N_{\mu-1}(X_u^+(w)) \cap X$. There is a cycle going through $\{x_1, w, x_2\}$ of length at most $2\mu \leq 2[(g - 4)/2] \leq g - 4$, contrary to the fact that the length of a shortest cycle in $G$ is equal to $g$.

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Similarly, we have  
\[ |N_{\mu-1}(N(u) - v - w) \cap X| \geq |N(u) - v - w|, \]
\[ |N_{\mu-1}(X^+_u(v)) \cap X| \geq |X^+_u(v)|, \]
\[ |N_{\mu-1}(X^+_u(w)) \cap X| \geq |X^+_u(w)|, \]
\[ |N_{\mu-1}(w) \cap X| \geq |X^+_u(w)|, \]
\[ |N_{\mu-1}(v) \cap X| \geq |X^+_u(v)|, \]
\[ |N_{\mu-1}(N(X^+_u(w)) - v) \cap X| \geq |X^+_u(w)|, \]
\[ |N_{\mu-1}(N(X^+_u(v)) - v) \cap X| \geq |X^+_u(v)|. \]

Likewise, the sets \( N_{\mu-1}(X^+_u(w)) \cap X, N_{\mu-1}(N(u) - v - w) \cap X, N_{\mu-1}(X^+_u(v)) \cap X, N_{\mu-1}(u) \cap X, N_{\mu-1}(v) \cap X, N_{\mu-1}(N(X^+_u(w)) - w) \cap X, \) and \( N_{\mu-1}(N(X^+_u(v)) - v) \cap X \) are pairwise disjoint. Hence we have

\[ \xi_3(G) \geq |X| \]
\[ \geq |N_{\mu-1}(X^+_u(w)) \cap X| + |N_{\mu-1}(v) \cap X| + |N_{\mu-1}(N(u) - v - w) \cap X| + |N_{\mu-1}(N(X^+_u(w)) - v) \cap X| + |N_{\mu-1}(N(X^+_u(v)) - w) \cap X| + |X_u^+(w)| + |X_u^+(v)| + |N(u) - v - w| + |X_u^-(v)| + |X_u^+(w)|
\]
\[ = d(u) + d(w) + d(v) - 4 \geq \xi_3(G). \]

Thus, the above inequalities become equalities, yielding

\[ X = (N_{\mu-1}(X^+_u(w)) \cap X) \cup (N_{\mu-1}(N(u) - v - w) \cap X) \cup (N_{\mu-1}(X^+_u(v)) \cap X) \cup (N_{\mu-1}(N(X^+_u(w)) - w) \cap X) \cup (N_{\mu-1}(N(X^+_u(v)) - v) \cap X). \]  

(1)

And

\[ |N_{\mu-1}(N(u) - v - w) \cap X| = |N(u) - v - w|; \]
\[ |N_{\mu-1}(N(X^+_u(w)) - w) \cap X| = |N(X^+_u(w)) - w| = |X_u^+(w)|; \]
\[ |N_{\mu-1}(N(X^+_u(v)) - v) \cap X| = |N(X^+_u(v)) - v| = |X_u^+(v)|. \]  

(2)

From (2) it follows that if \( |X_u^+(w)| > 0 \), then every vertex \( y \in X_u^+(w) \) has degree 2, which contradicts the fact that \( \delta \geq 3 \). Then \( X_u^+(w) = \emptyset \). Similarly, \( X_u^+(v) = \emptyset \). Furthermore, (2) also implies that each vertex \( x \in N(u) - v - w \) has one unique neighbor in \( X \) at distance \( \mu - 1 \), that is, \( |X_u^-(x)| = 1 \). Similarly, for edge \( uv \) we obtain that \( X_u^-(x) = \emptyset \), which implies that \( X_u^-(x) \neq \emptyset \) because \( \delta \geq 3 \). Take a vertex \( x' \in X_u^-(x) \), from (1) we conclude that there is a cycle passing through \( \{x', x, u\} \) and the vertex \( y \in N_{\mu-1}(x') \cap X \) of length at most \( 2(\mu - 1) + 4 \leq g - 1 \), then there would be a cycle of length less than \( g \), a contradiction.

Claim 2. \( \mu \geq \left( \frac{(g - 4)}{2} \right) \).

By contradiction, suppose that \( \mu \leq \left( \frac{(g - 4)}{2} \right) - 1 \). From Claim 1 we know there is an edge \( uv \) in \( C \) such that \( d(\{u, v\}, X) = \mu \). In this case, \( X_u^+(v) = X_u^+(u) = \emptyset \). Then \( C \) has a 2-path \( uuv \) such that \( d(w, X) = \mu \) or \( d(w, X) = \mu + 1 \).

Firstly, assume that \( d(w, X) = \mu \). Thus we have \( X_u^+(w) = \emptyset \). Arguing as in Claim 1 we have \( |N_{\mu}(X_u^-(w)) \cap X| \geq |X_u^-(w)| \) and \( |N_{\mu}(w) \cap X| \geq |X_u^-(w)| \). Furthermore, the sets \( N_{\mu}(X_u^-(w)) \cap X, N_{\mu}(w) \cap X, N_{\mu}(X_u^-(u)) \cap X, N_{\mu}(u) \cap X, N_{\mu}(w) \cap X \) and \( N_{\mu}(w) \cap X \) are pairwise disjoint. Therefore we have

\[ \xi_3(G) \geq |X| \]
\[ \geq |N_{\mu}(X_u^-(w)) \cap X| + |N_{\mu}(w) \cap X| + |N_{\mu}(X_u^-(u)) \cap X| + |N_{\mu}(u) \cap X| + |N_{\mu}(w) \cap X| + |N_{\mu}(w) \cap X| + |X_u^-(v)| + |X_u^-(w)| + |X_u^+(v)| + |X_u^+(w)|
\]
\[ = d(u) + d(w) + d(v) - 4 \geq \xi_3(G). \]

Thus, the above inequalities become equalities, yielding

\[ X = (N_{\mu}(X_u^-(w)) \cap X) \cup (N_{\mu}(X_u^-(u)) \cap X) \cup (N_{\mu}(X_u^-(w)) \cap X) \cup (N_{\mu}(X_u^-(v)) \cap X) \cup (N_{\mu}(X_u^-(u)) \cap X) \cup (N_{\mu}(X_u^-(v)) \cap X) \cap (N_{\mu}(w) \cap X). \]

(3)

and

\[ |N_{\mu}(X_u^-(w)) \cap X| = |X_u^-(w)|; \]
\[ |N_{\mu}(X_u^-(u)) \cap X| = |X_u^-(u)|; \]
\[ |N_{\mu}(X_u^-(v)) \cap X| = |X_u^-(v)|. \]  

(4)

From (4) we know that every vertex \( z \in X_u^-(w) \cup X_u^-(u) \cup X_u^-(v) \) has a unique neighbor at distance \( \mu \) in \( X \). As \( \delta \geq 3 \), there exists a vertex \( z' \in N(z) \cap N_{\mu}(X) \) and \( z' \in \{u, v, w\} \), for every \( z \in X_u^-(w) \cup X_u^-(u) \cup X_u^-(v) \). From (3) it follows that there is a cycle of length at most \( 2\mu + 5 \leq g - 1 \), contrary to the fact that the length of a shortest cycle in \( G \) is equal to \( g \).

Secondly if \( d(w, X) = \mu + 1 \), then it is analogous to the case of \( d(w, X) = \mu \).
As a consequence of both Claim 1 and Claim 2 we conclude that there exists an edge $uv$ in $C$ such that $d(u, v) \geq \lceil (g - 4)/2 \rceil$.

(2) Suppose now that $\mu = (g - 5)/2$ otherwise by item (1) we are done. And we denote $C_X = \{u \in V(C) : d(u, X) = (g - 5)/2\}$. By item (1) we can take an edge $uv$ in $G[C_X]$.

Firstly, assume $(N(u) - v) \cap C_X \neq \emptyset$ or $(N(v) - u) \cap C_X \neq \emptyset$. say, $(N(u) - v) \cap C_X \neq \emptyset$. Notice that $X_v^+(u) = X_v^+(w) = X_v^+(v) = \emptyset$ and that the sets $X_v^-(u), X_v^+(u), X_v^+(w), X_v^-(w), X_v^+(v)$ and $X_v^-(v)$ are pairwise disjoint. We will prove it by contradiction.

By contradiction, suppose that any vertex $u$ in $C_X$ satisfies $|N(g-5)/2(u) \cap X| \geq 2$. Then we have $|N(g-5)/2(X_v^-(u)) \cap X| \geq 2$, $|X_v^-(u)|, |N(g-5)/2(X_v^-(u)) \cap X| \geq 2|X_v^-(u)|$, and $|N(g-5)/2(X_v^-(u)) \cap X| \geq 2|X_v^-(u)|$. Since the sets $N(g-5)/2(X_v^-(u)) \cap X, N(g-7)/2(X_v^-(u)) \cap X, N(g-5)/2(X_v^+(u)) \cap X$, $N(g-7)/2(X_v^+(u)) \cap X, N(g-5)/2(X_v^+(w)) \cap X$ and $N(g-7)/2(X_v^+(w)) \cap X$ are pairwise disjoint, it follows that

$$\xi_3(G) \geq |X| \geq |N(g-5)/2(X_v^-(u)) \cap X| + |N(g-7)/2(X_v^-(u)) \cap X| + |N(g-5)/2(X_v^+(u)) \cap X| + |N(g-7)/2(X_v^+(u)) \cap X| + |N(g-5)/2(X_v^+(w)) \cap X| + |N(g-7)/2(X_v^+(w)) \cap X| \geq 2|X_v^-(u)| = |X_v^-(u)| + 2|X_v^-(w)| + |X_v^+(u)| + |X_v^+(w)| \geq \xi_3(G) + |X_v^-(u)| + |X_v^-(w)| + |X_v^+(u)| + |X_v^+(w)|. $$

Then $X_v^-(u) = X_v^+(w) = X_v^+(w)$ = $\emptyset$ and

$$X = (N(g-5)/2(u) \cap X) \cup (N(g-5)/2(v) \cap X) \cup (N(g-5)/2(w) \cap X). $$

Furthermore, we can obtain $|N(g-5)/2(u) \cap X| = |X_v^-(u)|, |N(g-5)/2(v) \cap X| = |X_v^+(w)|$ and $|N(g-5)/2(w) \cap X| = |X_v^+(w)|$. This means that $\mu = (g - 5)/2 \geq 2$. As $\delta \geq 3$, we have $|N(z) \cap (C_X - u)| \geq d(z) - 2 \geq 1$ for all $z \in V(C_X - u)$ (Otherwise a cycle of length at most $g - 2$ would appear). Take a vertex $z \in V(C_X - u)$ and consider a vertex $z' \in N(z) \cap (C_X - u)$. Then from (5) a cycle of length at most $g - 1$ would appear, a contradiction.

Secondly, if $(N(u) - v) \cap C_X = \emptyset$ and $(N(v) - u) \cap C_X = \emptyset$, then take a vertex $w$ in $N(v)$ with $d(w, X) = (g - 7)/2$. Hence $uvw$ is a 2-path in $C$, it is analogous to the above case.

Let $G = (V, E)$ be a $\lambda_3$-connected graph. An arbitrary $\lambda_3$-cut $F$ can be denoted by $[V(C), V(C')]$, where $C'$ and $C$ are the only two components of $G - F$. There are $X \subseteq V(C)$ and $Y \subseteq V(C')$ such that $X \cup Y$ is the set of the end vertices of $[V(C), V(C')]$, and so $[V(C), V(C')] = [X, Y]$.

A $\lambda_3$-connected graph $G$ is said to be $\text{super-}\lambda_3$, if $G$ is $\lambda_3$-optimal and every minimum 3-restricted edge cut isolates a component with exactly three vertices. A $\kappa_3$-connected graph $G$ is said to be $\text{super-}\kappa_3$, if $\kappa_3(G) = \xi_3(G)$ and the deletion of each minimum 3-restricted cut isolates a component with exactly three vertices.

**Lemma 2.2.** Let $G$ be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Let $[V(C), V(C')] = [X, Y]$ be a $\lambda_3$-cut. Then the following assertions hold:

(1) If $V(C) = X$, then $G$ is $\text{super-}\lambda_3$.

(2) If $G$ is not $\text{super-}\lambda_3$, then $C - X$ has a component with at least three vertices.

**Proof.** Since $g \geq 6$ and $\delta \geq 3$, by Theorem 1.1 $G$ is $\lambda_3$-connected.

(1) Suppose that $V(C) = X$, then each vertex of $C$ is incident with some edges of $[X, Y]$. If $|V(C)| = 3$, then we are done. So assume that $|V(C)| \geq 4$. Let $uvw$ be a 2-path of $C$. Because $\delta \geq 3$, we assume that $|X_v^-(u)| \geq 1$. Since girth $g \geq 6$, thus arguing as before, we have

$$\xi_3(G) \geq \lambda_3(G) = [\|u, Y\| + \|v, Y\| + \|w, Y\| + \|X_v^-(u), Y\| + \|X_v^+(w), Y\| + \|X_v^+(u), Y\| + \|X_v^-(w), Y\| + \|X_v^+(w), Y\| \geq 3 + d(u) - 1 + d(v) - 2 + d(w) - 1 > \xi_3(G),$$

which is a contradiction.

(2) By item (1) we have $C - X \neq \emptyset$. Suppose that any component of $C - X$ has at most two vertices. Let $C_1, C_2, \ldots, C_k$ be the components of $C - X$.

**Case 1.** Each component $C_i$ satisfies $|C_i| = 1$.

Take $C_1$ from $C_1, C_2, \ldots, C_k$. Let $C_1 = \{v\}$. Then $N(v) \subseteq X$. And $\delta \geq 3$, we pick $u, w \in N(v)$, and thus $uvw$ is a 2-path in $C$. Arguing as item (1),
we have
\[ \xi_3(G) \geq \lambda_3(G) = |X, Y| \]
\[ \geq |(N(u) - v, Y)| + |(N(v) - u - w, Y)| + |(N(v) - u - w, Y)| \]
\[ \geq |(N(u) - v)| + |N(v) - v| + |N(v) - u - w| \]
\[ = d(u) + d(v) + d(w) - 4 \geq \xi_3(G). \]

It follows that \[ |(N(u) - v, Y)| = |N(u) - v|, |(N(v) - u - w, Y)| = |N(v) - u - w|, |(N(v) - u - w)| = |(N(w) - v)|, \] and \[ X = (N(u) - v) \cup (N(v) - u - w) \cup (N(u) - v) \cup (N(v) - u - w) \cup (N(u) - v) \cup (N(v) - u - w). \] Hence \[ \{u, v\}, Y = \emptyset, \] which is a contradiction.

Case 2. There is a component \( C_1 \) with \( |C_1| = 2 \).
Assume that \( V(C_1) = \{u, v\} \). Then \( C_1 = K_2, \) and \( N(u) - v \subseteq X, N(v) - u \subseteq X \). Take \( w \in X \cap (N(v) - u) \). Then \( uvw \) is a 2-path in \( C \). As \( g \geq 6 \), arguing as in (1), we have
\[ \xi_3(G) \geq \lambda_3(G) = |X, Y| \]
\[ \geq |(N(u) - v, Y)| + |(N(v) - u - w, Y)| + |(N(w) - v) \cap X, Y| + |(w, Y)| \]
\[ = d(u) + d(v) + d(w) - 4 \geq \xi_3(G). \]

It follows that \[ |(N(u) - v, Y)| = |N(u) - v|, |(N(v) - u - w, Y)| = |N(v) - u - w|, |(N(v) - u - w)| = |(N(w) - v)|, \] and \[ X = (N(u) - v) \cup (N(v) - u - w) \cup (N(w) - v) \cup \{w\} \cup \{u, v\}. \] Therefore, for any \( x \in (N(u) - v) \cup (N(v) - u - w) \cup (N(w) - v) \cap X \), we have \[ |(x, Y) \cap X| = 1. \] Since \( g \geq 6 \) and \( \delta \geq 3 \), it follows that \[ N(x) \cap (X - x) = \emptyset. \] So \( x \) is adjacent to some \( C_i \)'s (\( 2 \leq i \leq k \)). If there is a \( C_1 = \{y\} \) such that \( y \in N(x) \), then \( N(y) \subseteq X \). As \( g \geq 6 \) and \( \delta \geq 3 \), we have \[ |N(y) \cap (N(v) - u)| \leq 1, |N(y) \cap (N(v) - u)| \leq 1 \] and \[ |N(y) \cap (N(w) \cap X)| \leq 1. \]

Without loss of generality, we assume that \[ |N(y) \cap (N(w) \cap X)| = 1, \] then \( N(y) \cap (N(v) - u) = \emptyset, \) and \( u, v \} \in (N(y) \cap (N(u) - v) \} \geq 2. \] There is a cycle with length smaller than \( g \), a contradiction. If \[ |N(y) \cap (N(w) \cap X)| = 0, \] then \[ |N(y) \cap (N(v) - u)| \geq 2. \] There is also a cycle of length smaller than \( g \), which is impossible.

If there is a \( C_2 = 2 \) which \( x \) is adjacent to, then it is analogous to the case of \( |C_1| = 1 \). We discuss the neighbors of each vertex in \( C_j \), we can obtain the required result.

Recall that in the line graph \( L(G) \) of a graph \( G \), each vertex represents an edge of \( G \), and two vertices in a line graph are adjacent if and only if the corresponding edges of \( G \) are adjacent. Let us consider the edges \( x_1y_1, x_2y_2 \in E(G) \). The distance between the corresponding vertices of \( L(G) \) satisfies
\[ d_{L(G)}(x_1y_1, x_2y_2) = d_G(\{x_1, y_1\}, \{x_2, y_2\}) + 1, \]
which is useful to prove that
\[ D(G) - 1 \leq D(L(G)) \leq D(G) + 1. \]

3 Some sufficient conditions for graphs to be super-\( \lambda_3 \) (resp. super-\( \kappa_3 \))

Now, we will show Theorem 3.1 by contradiction.

Theorem 3.1. Let \( G \) be a connected graph with girth \( g \geq 4 \) and minimum degree \( \delta \geq 3 \). The following assertions hold:

- (1) If \( D(G) \leq g - 4 \), then \( G \) is super-\( \lambda_3 \).
- (2) If \( D(G) \leq g - 5 \), then \( G \) is super-\( \kappa_3 \).
- (3) If the diameter of the line graph \( D(L(G)) \leq g - 4 \), then \( G \) is super-\( \lambda_3 \).
- (4) If the diameter of the line graph \( D(L(G)) \leq g - 5 \), then \( G \) is super-\( \kappa_3 \).

Proof. Since \( g \geq 4 \), clearly \( G \) is different from the graphs in Fig.1. Thus, by Theorem 1.1, \( G \) is \( \lambda_3 \)-connected. Moreover, if \( g \in \{4, 5, 6\} \), then theorem clearly holds. So we assume that \( g \geq 7 \). By part (2) of Theorem 1.2, \( G \) is \( \kappa_3 \)-connected.

(1) From Theorem 1.2 it follows that \( \lambda_3 = \xi_3 \).
Assume that \( G \) is not super-\( \lambda_3 \). Let \( V(C), V(C') = \{X, Y\} \) be a \( \lambda_3 \)-cut with \( |V(C)| \geq 4, |V(C')| \geq 4 \). By Lemma 2.2 we know that both \( C - X \) and \( C - Y \) contain a connected component say \( H \) and \( K \), respectively, of cardinality at least three vertices. Hence both \( X \) and \( Y \) are cutsets with \( |X|, |Y| \leq \xi_3(G) \). From Lemma 2.1 there exist two vertices \( u \in V(H) \) and \( \pi \in V(K) \) such that \( g - 4 \geq D(G) \geq d(u, \pi) \geq d(u, X) + 1 + d(\pi, Y) \geq 2[(g - 4)/2] + 1 \), which is a contradiction if \( g \) is even.

And for \( g \) odd all the inequalities become equalities. This means that \( g = (g - 5)/2 \) and \( \max\{d(u, X) : u \in V(H)\} = (g - 5)/2 \). Thus by Lemma 2.1, we can find \( u \in V(H) \) with \( d(u, X) = (g - 5)/2 \) such that \( N((g - 5)/2) \cap X = \{x\} \) for some \( x \in X \); and we can find \( \pi \in V(K) \) with \( d(\pi, Y) = (g - 5)/2 \) such that \( N((g - 5)/2) \cap Y = \{\pi\} \) for some \( \pi \in Y \). As \( d(u, \pi) = g - 4 \), it follows that \( x, \pi \in [X, Y] \). Clearly we can find a vertex \( v \in N(u) \) with \( d(v, X) = (g - 5)/2 \), because otherwise \( |N((g - 5)/2) \cap X| \geq |N(u)| \geq 2. \) Since \( d(v, \pi) = g - 4 \) we must have \( x \in N((g - 5)/2)(v) \) or \( \pi \in N((g - 3)/2) \). As a consequence, the path from \( u \) to \( \pi \) together with the path
from $v$ to $\pi$ and the edge $uv$ form a cycle of length at most $g - 2$, which is a contradiction.

(2) From Theorem 1.2 it follows that $\kappa_3 = \xi_3$. Assume that $G$ is not super-$\kappa_3$. Let $X$ be an any $\kappa_3$-cut and consider two connected components $C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4$, $|V(\overline{C})| \geq 4$. From Lemma 2.1 there exist two vertices $u \in V(C)$ and $\pi \in V(\overline{C})$ such that $g - 5 \geq D(G) \geq d(u, \pi) \geq d(u, X) + d(\pi, X) \geq 2[(g - 4)/2]$, which is a contradiction if $g$ is even.

And for $g$ odd all the inequalities become equalities. This means that max \{d(u, X) : u \in V(C)\} = (g-5)/2 and max \{d(\pi, Y) : \pi \in V(\overline{C})\} = (g-5)/2. Thus by Lemma 2.1, we can find $u \in V(C)$ with $d(u, X) = (g - 5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\pi \in V(\overline{C})$ with $d(\pi, Y) = (g - 5)/2$ such that $N_{(g-5)/2}(\pi) \cap Y = \{\pi\}$ for some $\pi \in Y$. As $d(u, \pi) = g - 5$, it follows that $x = \pi$. Clearly we can find a vertex $v \in N(u)$ with $d(v, X) = (g - 5)/2$. Since $d(v, \pi) = g - 5$ we must have $x \in N_{(g-5)/2}(v)$. As a consequence, the path from $u$ to $x$ together with the path from $v$ to $x$ and the edge $uv$ form a cycle of length at most $g - 4$, which is a contradiction.

(3) Since $D(L(G)) \leq g - 4$, then the diameter $D(G) \leq g - 3$, which means that $\lambda_3 = \xi_3$ by Theorem 1.2. Assume that $G$ is not super-$\lambda_3$. Let $[V(C), V(\overline{C})] = [X, Y]$ be a $\lambda_3$-cut with $|V(C)| \geq 4$, $|V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C - X$ and $\overline{C} - Y$ contain a connected component say $H$ and $K$, respectively, of cardinality at least three. Hence both $X$ and $Y$ are cutsets with $|X|, |Y| \leq \xi_3(G)$. From Lemma 2.1 there exists an edge $uv$ in $C - X$ and there exist an edge $\pi \psi$ in $\overline{C} - Y$ satisfying $d(u, v, X) \geq [(g - 4)/2]$ and $d(\pi, \psi, Y) \geq [(g - 4)/2]$. Then by using (6) we have

$$g - 4 \geq D(L(G)) \geq d_{L(G)}(uv, \pi \psi) = d_G(\{u, v\}, \{\pi, \psi\}) + 1 \geq d_G(\{u, v\}, X) + 1 + d_G(Y, \{\pi, \psi\}) + 1 \geq 2[(g - 4)/2] + 2,$$

which is impossible.

(4) Now $D(L(G)) \leq g - 5$. Thus the diameter $D(G) \leq g - 4$, which means that $\kappa_3 = \xi_3$ by Theorem 1.2. Assume that $G$ is not super-$\kappa_3$. Let $X$ be an any $\kappa_3$-cut and consider two connected components $C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4$, $|V(\overline{C})| \geq 4$. From Lemma 2.1 there exists an edge $uv$ in $C - X$ and there exists an edge $\pi \psi$ in $\overline{C} - Y$ satisfying $d(u, v, X) \geq [(g - 4)/2]$ and $d(\pi, \psi, X) \geq [(g - 4)/2]$. Then by using (6) we have

$$g - 5 \geq D(L(G)) \geq d_{L(G)}(uv, \pi \psi) = d_G(\{u, v\}, \{\pi, \psi\}) + 1 \geq d_G(\{u, v\}, X) + d_G(X, \{\pi, \psi\}) + 1 \geq 2[(g - 4)/2] + 1,$$

which is impossible. 

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