A Flexible Generalized Minimum Variance Controller for MIMO Time-Varying Systems

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Abstract: - A generalized minimum variance controller is developed for servo application for multiple input and multiple output systems having time-varying dynamics. The plant to be controlled is a controlled autoregressive moving average model and the control objective is to minimize a generalized minimum variance performance index that is an extension of the standard index from linear time invariant cases for linear time-varying applications.

Key-Words: - Multiple variable systems, generalized minimum variance control, time-varying systems.

1 Introduction

The multiple input and multiple output (MIMO) generalized minimum variance controller (GMVC) was developed by Koivo [1] for linear time-invariant (LTI) systems extending the GMVCs from LTI single input and single output (SISO) plants [2], [3] for MIMO systems. The GMVCs are popular controllers for adaptive control and have been extended from transfer function framework to state space models for LTI systems [4]. They have also been applied for DC motor control in order to replace the classical PID control for optimal control for industrial applications [5-7]. The LTI MIMO GMVC was developed using a pseudocommutation technique for overcoming noncommutativity of MIMO transfer functions with respect to multiplication. The LTI MIMO GMVC is very flexible and has all the standard features of an LTI SISO GMVC. The performance index of the GMVC includes both a variance of tracking error from a filtered plant output and a quadratic function of filtered input. It compromises between the tracking error and the fluctuation of the input.

The standard LTI SISO GMVC [2], [3] has also been extended from LTI SISO plants for linear time-varying (LTV) SISO systems based on a different pseudocommutation technique specifically developed for an LTV SISO GMVC [8]. LTV GMVCs were developed recently for LTV MIMO systems without using any pseudocommutation [9]. However, it uses autoregressive LTV filters for optimization rather than moving average filters that are used by the standard LTI GMVCs. As a result, the difference of plant input/output cannot be included into the cost functional directly although they are very useful in applications. In this paper LTV moving average filters in general forms are introduced for optimization extending finally the standard LTI GMVC from LTI MIMO systems for LTV MIMO plants.

2 Control Objective

Having p inputs and p outputs the systems to be controlled is represented using the following LTV MIMO CARMA model.

$$A(k,q^{-1})Y(k+d) = B(k,q^{-1})U(k) + C(k,q^{-1})W(k+d) \tag{1}$$

where $U(k)$ and $Y(k)$ are $p\times 1$ plant input and output vector, d is a positive integer representing the time delay between them, $W(k)$ is a $p\times 1$ zero mean, independent Gaussian noise, whose variance is a uniformly bounded $p\times p$ time-varying matrix and q^{-1} is the one-step-delay operator such that for a product of time-varying matrices $F(k)G(k-1)$ we have

$$q^{-1}F(k)G(k-1)=F(k-1)q^{-1}G(k-1)=F(k-1)G(k-2) \tag{2}$$

In CARMA model (1)

$$A(k,q^{-1})=I+A_1(k)q^{-1}+A_2(k)q^{-2}+\ldots+A_n(k)q^{-n}$$
$$B(k,q^{-1})=B_0(k)+B_1(k)q^{-1}+\ldots+B_m(k)q^{-m}$$

$$C(k,q^{-1})=I+C_1(k)q^{-1}+\ldots+C_h(k)q^{-h} \tag{3}$$

are $p\times p$ LTV moving average operators (MAO's) defined using matrix polynomials in the delay operator. The LTV coefficient matrices of the polynomial, $A_i(k)$, $B_j(k)$ and $C_r(k)$, $i=1, 2, \ldots, n$, $j=0, 1, \ldots, m$, $r=1, 2, \ldots, h$, are uniformly bounded $p\times p$ time-varying matrices. Similar to the LTI case, the determinant of $B_0(k)$ is assumed to be uniformly bounded away from zero such that the system has a unique and time-invariant delay. The inverse operation of an LTV MAO is defined as an LTV autoregressive operator (ARO) that is denoted as $A^{-1}(k,q^{-1})$ [8]. It is assumed that both $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable.

Given a $p\times 1$ uniformly bounded reference $Z(k)$ and the input and output data up to and including the current time k the generalised minimum variance control objective is to minimise the following quadratic GMVC performance cost index.

$$\begin{aligned}J(k+d)=E\{[(\Psi(k+d)-S(k+d))^T\Lambda(k)(\Psi(k+d)\\ -S(k+d))\\ +(R(k,q^{-1})U(k))^T V(k)R(k,q^{-1})U(k)]/D(k)\}\end{aligned} \tag{4}$$

where the superscript T is for matrix transpose and E is for mathematical expectation conditioned on the data set $D(k)=\{Y(k), U(k), Y(k-1), U(k-1), \ldots\}$. In the above cost functional, $\Lambda(k)$ and $V(k)$ are time-varying weighting matrices for variance of the generalized output tracking error and quadratic function of the filtered input vector. Both are assumed to be uniformly bounded and uniformly positive definite. This assumption should be nonrestrictive because the choice of both is in our hands. The generalized output and reference are defined using LTV MIMO MAO filters as follows.

$$\begin{aligned}\Psi(k+d)&=P(k,q^{-1})y(k+d),\\ S(k+d)&=Q(k,q^{-1})Z(k+d)\end{aligned} \tag{5}$$

The LTV MIMO weighting filters have the forms

$$\begin{aligned}P(k,q^{-1})&=I+P_1(k)q^{-1}+P_2(k)q^{-2}+\ldots+P_{np}(k)q^{-np}\\ Q(k,q^{-1})&=Q_0(k)+Q_1(k)q^{-1}+\ldots+Q_{nq}(k)q^{-nq}\\ R(k,q^{-1})&=R_0(k)+R_1(k)q^{-1}+\ldots+R_{nr}(k)q^{-np}\end{aligned} \tag{6}$$

where all the time-varying coefficient matrices of the filters should be chosen to be uniformly bounded and the determinate of $Q_0(k)$ and $R_0(k)$ should be chosen to be uniformly bounded away from zero. In addition we choose $P^{-1}(k,q^{-1})$ and $Q^{-1}(k,q^{-1})$ to be exponentially stable LTV AROs.

3 GMVC

We first develop a predictor (MVP) for the LTV CARMA model in terms of the filtered output in order to deal with the time delay. Left multiplying $P(k,q^{-1})A^{-1}(k,q^{-1})$ on both sides of the LTV CARMA model (1) we have

$$\begin{aligned}P(k,q^{-1})Y(k+d)=P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k)\\ +P(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})W(k+d)\end{aligned} \tag{7}$$

The division of LTV MAO's in the last term of the above equation can be written in the form of long division as follows.

$$\begin{aligned}A^{-1}(k,q^{-1})C(k,q^{-1})&=F(k,q^{-1})\\ &+A^{-1}(k,q^{-1})G(k,q^{-1})q^{-d}\end{aligned} \tag{8}$$

where

$$\begin{aligned}F(k,q^{-1})&=1+F_1(k)q^{-1}+F_2(k)q^{-2}+\\ &\ldots+F_{d-1}(k)q^{-d+1}\end{aligned} \tag{9}$$

and

$$G(k,q^{-1})=G_0(k)+G_1(k)q^{-1}+\ldots+G_s(k)q^{-s} \tag{10}$$

Substituting (8) into (7) and noting (5) it follows that

$$\begin{aligned}\Psi(k+d)&=P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k)\\ &+P(k,q^{-1})F(k,q^{-1})W(k+d)\\ &+P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})W(k)\end{aligned} \tag{11}$$

Letting

$$P(k,q^{-1})F(k,q^{-1})=H(k,q^{-1})+L(k,q^{-1})q^{-d} \tag{12}$$

where

$$H(k,q^{-1}) = 1 + H_1(k)q^{-1} + H_2(k)q^{-2} + \dots + H_{d-1}(k)q^{-d+1} \tag{13}$$

has all the terms that have delays between zero and *-d+1* inclusive. Substituting (12) into (11) we have

$$\begin{aligned}\Psi(k+d) &= P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k) \\ &+ H(k,q^{-1})W(k+d) + [L(k,q^{-1}) \\ &+ P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]W(k)\end{aligned} \tag{14}$$

Taking mathematical expectation conditioned on *D(k)* and substituting (1) into the above equation we have minimum variance prediction of the filtered output as follows.

$$\begin{aligned}\hat{\Psi}(k+d) &= P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k) \\ &+ [L(k,q^{-1}) + P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]W(k) \\ &= P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k) + [L(k,q^{-1}) \\ &+ P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]C^{-1}(k-d,q^{-1}) \\ &[A(k-d,q^{-1})Y(k) - B(k-d,q^{-1})U(k-d)]\end{aligned} \tag{15}$$

where $\hat{\Psi}(k+d) = E[\Psi(k+d)/D(k)]$ is the *d*-step-ahead prediction of $\Psi(k)$.

3.1 GMVC Theorem

If the LTV AROs $A^{-1}(k,q^{-1})$, $C^{-1}(k,q^{-1})$ and $P^{-1}(k,q^{-1})$ are exponentially stable, the LTV MIMO GMVC for the CARMA model (1) is given by

$$\begin{aligned}\hat{W}(k) &= C^{-1}(k-d,q^{-1})[A(k-d,q^{-1})Y(k) \\ &- B(k-d,q^{-1})U(k-d)]\end{aligned} \tag{16}$$

$$\begin{aligned}U(k) &= T^{-1}(k,q^{-1})\{A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})Z(k+d) \\ &- [G(k,q^{-1}) + A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})]\hat{W}(k)\}\end{aligned} \tag{17}$$

Where

$$\begin{aligned}T(k,q^{-1}) &= A(k,q^{-1})P^{-1}(k,q^{-1})\Lambda^{-1}(k) \\ &B_0^{-T}(k)R_0^T(k)V(k)R(k,q^{-1}) + B(k,q^{-1})\end{aligned}. \tag{18}$$

3.2 Proof

Compare (14) and (15) we have

$$\hat{\Psi}(k+d) = \Psi(k+d) - H(k,q^{-1})W(k+d) \tag{19}$$

Noting cost index (4) we have

$$\begin{aligned}J(k+d) &= E\{[(\hat{\Psi}(k+d) - S(k+d))^T \Lambda(k) \\ &(\hat{\Psi}(k+d) - S(k+d)) + (R(k,q^{-1}) \\ &U(k))^T V(k) R(k,q^{-1})U(k) \\ &+ (H(k,q^{-1})W(k+d))^T \Lambda(k) \\ &H(k,q^{-1})W(k+d)]/D(k)\}\end{aligned} \tag{20}$$

It follows form (1) and (20) that

$$\begin{aligned}\frac{\partial J(k+d)}{\partial U(k)} &= 2B_0^T(k)\Lambda(k)[\hat{\Psi}(k+d) - S(k+d)] \\ &+ 2R_0^T(k)V(k)[R(k,q^{-1})U(k)]\end{aligned} \tag{21}$$

and

$$\frac{\partial^2 J(k+d)}{\partial U^2(k)} = 2B_0^T(k)\Lambda(k)B_0(k) + 2R_0^T(k)V(k)R_0(k) \tag{22}$$

Bedause both *Λ(k)* and *V(k)* are uniformly positive definite and both $B_0(k)$ and $R_0(k)$ are uniformly nonsingular the above equation shows that there exists optimal control *U(k)* such that the generalized minimum variance cost index will be achieved. Letting (21) zero the LTV MIMO GMVC can be determined as follows.

$$\begin{aligned}&\Lambda^{-1}(k)B_0^{-T}(k)R_0^T(k)V(k)R(k,q^{-1})U(k) = \\ &S(k+d) - P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})U(k) \\ &- P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})W(k) \\ &- L(k,q^{-1})W(k)\end{aligned} \tag{23}$$

Left multiplying $A(k,q^{-1})P^{-1}(k,q^{-1})$ on both sides of the above equation, solve for *U(k)* and substitute (1) we have the GMVC,

$$\begin{aligned}&[A(k,q^{-1})P^{-1}(k,q^{-1})\Lambda^{-1}(k)B_0^{-T}(k)R_0^T(k)V(k)R(k,q^{-1}) \\ &+ B(k,q^{-1})]U(k) \\ &= A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})Z(k+d) - [G(k,q^{-1}) \\ &+ A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})]C^{-1}(k-d,q^{-1}) \\ &[A(k-d,q^{-1})Y(k) - B(k-d,q^{-1})U(k-d)]\end{aligned} \tag{24}$$

Comparing (1) with (16) we have

$$C(k,q^{-1})\tilde{W}(k+d) = 0 \tag{25}$$

where

$$\tilde{W}(k+d) = W(k+d) - \hat{W}(k+d) \tag{26}$$

is the noise estimation error of (16). Because of the exponential stability of $C^{-1}(k,q^{-1})$ this error will always decay exponentially to zero regardless any bounded initial condition. As a result, *U(k)* will also converge exponentially to the optimal control. Noting (1), (17), (23), (24) and (25) we have the closed loop equation for the LTV GMVC

$$\begin{bmatrix} C(k-d,q^{-1}) & 0 & 0 \\ -E(k,q^{-1}) & T(k,q^{-1}) & 0 \\ 0 & -B(k-d,q^{-1})q^{-d} & A(k-d,q^{-1}) \end{bmatrix} \begin{bmatrix} \tilde{W}(k) \\ U(k) \\ Y(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -E(k,q^{-1}) & A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1}) \\ C(k-d,q^{-1}) & 0 \end{bmatrix} \begin{bmatrix} W(k) \\ Z(k+d) \end{bmatrix} \tag{27}$$

where

$$E(k,q^{-1}) = G(k,q^{-1}) + A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1}) \tag{28}$$

The closed-loop stability is determined by the following LTV ARO.

$$M^{-1}(k,q^{-1}) = \begin{bmatrix} C(k-d,q^{-1}) & 0 & 0 \\ -E(k,q^{-1}) & T(k,q^{-1}) & 0 \\ 0 & -B(k-d,q^{-1})q^{-d} & A(k-d,q^{-1}) \end{bmatrix}^{-1} \tag{29}$$

Because $M(k,q^{-1})$ is a triangle matrix and both $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable the closed-loop system is exponentially stable if and only if $T^{-1}(k,q^{-1})$ is exponentially stable. According to (18) the exponential stability of $T^{-1}(k,q^{-1})$ is in our hands because it can be modified by choosing appropriate weighting matrices and filters.

4 Simulation

The plant to be controlled is an LTV 2I2O system as follows.

$$\begin{aligned} &Y(k+2) + A(k)Y(k+1) = U(k) \\ &+ B(k)U(k-1) + W(k+2) + C(k)W(k+1) \end{aligned} \tag{34}$$

where *W(k)* is a 2 dimensional independent Gaussian process with zero mean and identity variance, the time-varying parameter matrices are

$$A(k) = \begin{bmatrix} 0.3\cos(0.3k) & 0.2\sin(3k) \\ 0 & 0.2 sign[\cos(0.5k)] \end{bmatrix} \tag{35a}$$

$$B(k) = \begin{bmatrix} 1.1 & 0.2\cos(0.5k) \\ 0 & 0.3 sign[\sin(0.6k)] \end{bmatrix} \tag{35b}$$

$$C(k) = \begin{bmatrix} 0.5\dfrac{k+1}{k+3} & 1.5[1+\exp(-.1k)]\cos(0.5k) \\ 0 & 0.5\cos(0.1k) \end{bmatrix} \tag{35c}$$

Both $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable because both *A(k)* and *C(k)* are triangular matrices and the absolute values of all of their diagonal elements are uniformly less than one. However, $B^{-1}(k,q^{-1})$ is exponentially unstable because *B(k)* is a triangle matrix and one of its diagonal elements is a constant and greater than one. The weighting matrices are

$$\Lambda(k) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \qquad V(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \tag{36}$$

The output filter and reference filter are chosen as

$$P(k,q^{-1}) = I + P(k)q^{-1} = Q(k,q^{-1}) \tag{37}$$

where

$$P(k) = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}. \tag{39}$$

Here we use the same exponentially stable filters for the output and reference in order for the output to follow the reference. The input filter is

$$R(k,q^{-1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} q^{-1} \tag{40}$$

The reference and output are given in Fig. 1 and Fig. 2. They show that the LTV MIMO GMVC is able to drive both outputs to follow the references regardless the process noise and rapidly time-varying parameters in the CARMA model. The control variables are shown in Fig. 3.

5 Conclusion

A servo GMVC is developed for MIMO LTV systems without using pseudocommutation. The LTV GMVC is a natural extension of the standard LTI GMVC from LTI MIMO transfer functions for LTV MIMO transfer operators for optimal control of an LTV CARMA model. It uses LTV MAO weighting filters for flexibility and robustness of the closed-loop control systems and is applicable to a large class of LTV systems.

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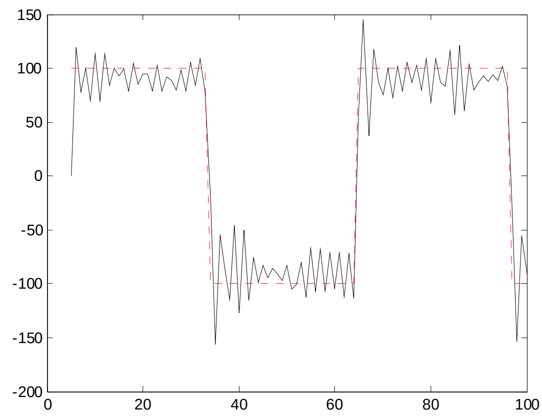

Fig. 1 The first output and its reference. The dotted line represents the reference and the solid line represent the real output.

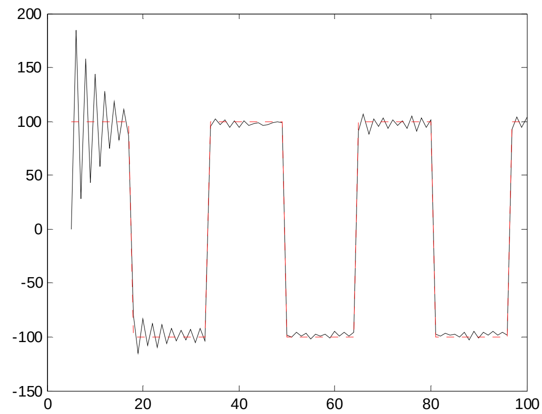

Fig. 2 The second output and its reference. The dotted line represents, the reference and the solid line represent the real output.

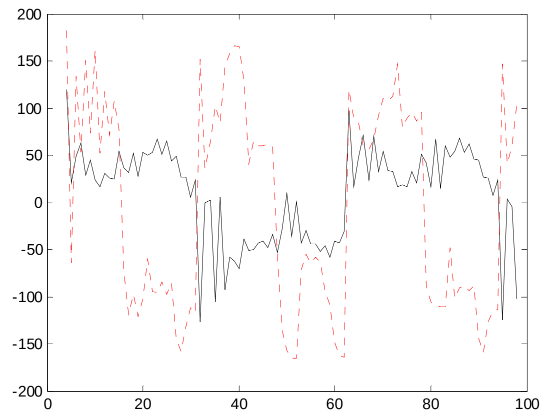

Fig. 3 Output of the GMVC. The solid line is the first plant input and the dotted line is the second plant input.