The Continuous-Time $\mathcal{H}_\infty$ Model Matching Problem: 1 DOF Static State Feedback with Integral Control Approach

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Abstract: The aim of this paper is to develop a new approach for a solution of the continuous-time model matching problem with a static state feedback in the sense of $\mathcal{H}_\infty$ optimality criterion. The main contribution is to reformulate the $\mathcal{H}_\infty$ model matching problem in linear matrix inequality settings, to present the solvability conditions and to give a design procedure for a one degree of freedom static state feedback with integral control law. The results are applied to an example system.

Key–Words: Model Matching Problem, Linear Matrix Inequalities, $\mathcal{H}_\infty$ Optimal Control Problem, One Degree of Freedom Static State Feedback, Integral Control.

1 Introduction

The model matching problem is one of the most familiar problems in the control theory [16]. The continuous-time $\mathcal{H}_\infty$ model matching problem (MMP) is to find a controller transfer matrix $R(s)$ which is stable and causal, that is $R(s) \in \mathbb{R}\mathcal{H}_\infty$, which minimizes the $\mathcal{H}_\infty$ norm of $G_m(s) - G(s)R(s)$ where $G_m(s)$ and $G(s)$ are the model and the system transfer matrices, respectively. Moreover, $G_m(s)$ and $G(s)$ are stable and proper transfer matrices. That is to say, the closed-loop performance $G(s)R(s)$ approximates the desired performance $G_m(s)$ such that,

$$\gamma_{opt} = \inf_{R(s) \in \mathbb{R}\mathcal{H}_\infty} \|G_m(s) - G(s)R(s)\|_\infty.$$ 

In the literature, there are some results on the $\mathcal{H}_\infty$ MMP: [6, 8, 9]. Moreover, the solutions of the continuous- and discrete-time $\mathcal{H}_\infty$ MMP via linear matrix inequality (LMI) optimization are given in [1, 2, 3, 4, 13]. However, in none of them, one degree of freedom static state feedback with integral control structure is used for feedback configuration.

In this study, a special formulation is developed to solve the continuous-time $\mathcal{H}_\infty$ MMP by a one degree of freedom (1 DOF) static state feedback with integral control. One degree of freedom controller means that there is only one controller block in the closed system, [14]. This formulation enables us to use the methods which are presented for the solution of the continuous-time $\mathcal{H}_\infty$ optimal control problem (OCP) and so the continuous-time $\mathcal{H}_\infty$ MMP can completely be solved by the LMI-based numerical optimization.

The paper is organized in the following way: In Section 2, a special formulation for the continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF static state feedback with integral control is presented in LMI. In Section 3, the main result is given by a theorem which provides have the existence conditions of the solution. In Section 4, the problem is examined for the strictly proper case. In Section 5, the 1 DOF static state feedback with integral control is constructed by using the synthesis theorem. A numerical example and the conclusions are finally given in Section 6 and 7, respectively.

Notations

- $\mathbb{R}$ The set of real numbers.
- $\mathbb{R}^{n \times m}$ The set of $n \times m$ real matrices.
- $I_n$ Identity matrix of $n \times n$ dimension.
- $0_{n \times m}$ The matrix which has $n \times m$ dimension, and all elements are zero.
- $\text{Ker}M$ The null space of the linear operator $M$.
- $\text{Im}M$ The range of the linear operator $M$. 

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\(N^T\) The transpose of the matrix \(N\).

\(P > 0\) The matrix \(P\) is positive definite.

\(\lambda_{\text{max}}(A)\) The maximal eigenvalue of the matrix \(A\).

\(\sigma_{\text{max}}(A)\) The maximal singular value of the matrix \(A\) which is defined

\[
\sigma_{\text{max}}(A) = \sqrt{\lambda_{\text{max}}(A^T A)}.
\]

\(\|G(s)\|_\infty\) The \(\mathcal{H}_\infty\) norm of the transfer matrix \(G(s)\) is defined as

\[
\|G(s)\|_\infty = \sup_{\omega \in [0, \infty]} \sigma_{\text{max}}[G(j\omega)].
\]

2 The Continuous-time \(\mathcal{H}_\infty\) MMP by a 1 DOF Static State Feedback with Integral Control in LMI Optimization

In order to solve the continuous-time \(\mathcal{H}_\infty\) MMP via LMI approach, the problem should be reformulated as the standard continuous-time \(\mathcal{H}_\infty\) OCP. First of all, I will take any state-space equations of the given system \(G(s)\) and the model system \(G_m(s)\) as follows:

\[
G(s): \quad \dot{x}(t) = Ax(t) + Bv(t) \quad (1)
\]
\[
y_s(t) = Cx(t) + Dv(t) \quad (2)
\]

\[
G_m(s): \quad \dot{q}(t) = Fq(t) + Gw(t) \quad (3)
\]
\[
y_m(t) = Hq(t) + Jw(t) \quad (4)
\]

where \(x(t) \in \mathbb{R}^n\), \(q(t) \in \mathbb{R}^m\); \(v(t), w(t), y_s(t)\) and \(y_m(t) \in \mathbb{R}^m\). The control input \(u(t)\) is generated by a static state feedback controller:

\[u(t) = Kx(t)\].

In Figure 1, the block diagram of a continuous-time \(\mathcal{H}_\infty\) MMP by a static state feedback with integral control is given. In this formulation the steady-state value of the output \(y_s(t)\) will follow a step function input with zero error. In this paper, a 1 DOF control structure is proposed, [14].

![Figure 1. The block diagram of model matching system with 1 DOF static state feedback in the integral control.](image-url)
As a result, the continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF static state feedback with integral control is equivalent to the continuous-time $\mathcal{H}_\infty$ OCP. Figure 2 shows this idea:

![Block Diagram](image)

Figure 2. The block diagram of the general form of $\mathcal{H}_\infty$ OCP with a static controller.

The closed-loop transfer matrix from $w(t)$ to $z(t)$ is

$$T_{zw}(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}$$

where

$$A_{cl} = A + B_2KC_2$$
$$B_{cl} = B_1$$
$$C_{cl} = C_1 + D_2KC_2$$
$$D_{cl} = D_1.$$ (16) (17) (18) (19)

If the matrix $K$ which makes stable the matrix $A + BK$, can be found out, it is said that the matrix pair $(A, B)$ is stabilizable.

The following lemma can be given for the internal stability of the closed-loop system:

**Lemma 1** For the system in (5), (6) and (7), there is a matrix $K$ such that the matrix $A_{cl} = A + B_2KC_2$ is Hurwitz if and only if the matrix pair

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \begin{bmatrix} B \\ -D \end{bmatrix}$$

is stabilizable and the matrix $F$ is Hurwitz.

**Proof:** When $A_i, B_2, C_2$ and $K$ are used in $A_{cl}$, the following relation is obtained:

$$A_{cl} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} K \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$ (20)

Therefore, the matrix $A_{cl}$ is Hurwitz if and only if the matrix

$$\begin{bmatrix} A + BK & B \\ -C - DK & -D \end{bmatrix}$$

and the matrix $F$ are Hurwitz. The matrix

$$\begin{bmatrix} A + BK & B \\ -C - DK & -D \end{bmatrix}$$

can be rewritten as

$$\begin{bmatrix} A + BK & B \\ -C - DK & -D \end{bmatrix} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} + \begin{bmatrix} B \\ -D \end{bmatrix} K \begin{bmatrix} I & 0 \end{bmatrix}.$$ (24)

If we take

$$L = K \begin{bmatrix} I \\ 0 \end{bmatrix}$$

, since the matrix $K$ can always be determined by

$$K = L \begin{bmatrix} I \\ 0 \end{bmatrix}$$

, the matrix

$$\begin{bmatrix} A + BK & B \\ -C - DK & -D \end{bmatrix}$$

is asymptotically stable if and only if the matrix pair

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \begin{bmatrix} B \\ -D \end{bmatrix}$$

is stabilizable. [15]

For a synthesis theorem on the LMI-based solution of the continuous-time $\mathcal{H}_\infty$ MMP with integral control, let us give the following lemmas. They will be used to prove the theorem which will be presented later. The first lemma is well known as the **Bounded Real Lemma** and can be used to turn the continuous-time $\mathcal{H}_\infty$ OCP into an LMI:

**Lemma 2** Consider a continuous-time transfer matrix $T(s)$ of (not necessarily minimal) realization

$$T(s) = D + C(sI - A)^{-1}B.$$ (29)

The following statements are equivalent:

**i)**

$$\|D + C(sI - A)^{-1}B\|_{\infty} < \gamma$$

and the matrix $A$ is Hurwitz,

**ii)** there is a solution $X > 0$ to the LMI:

$$\begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$ (31)
Proof: See [7]. ■

Lemma 3 Suppose $P$, $Q$ and $H$ are matrices and the matrix $H$ is symmetric. The matrices $N_P$ and $N_Q$ are full rank matrices satisfying $ImN_P = KerP$ and $ImN_Q = KerQ$. Then there is a matrix $J$ such that,

$$H + P^TJQ + Q^TP < 0$$  \hspace{1cm}  \text{(32)}

if and only if the inequalities

$$N_P^THN_P < 0 \quad \text{and} \quad N_Q^THN_Q < 0 \hspace{1cm} \text{(33)}$$

are both satisfied.

Proof: See [10]. ■

Lemma 4 The block matrix

$$\begin{bmatrix} P & M \\ MT & N \end{bmatrix} < 0$$  \hspace{1cm}  \text{(34)}

if and only if

$$N < 0 \quad \text{and} \quad P - MN^{-1}MT < 0.$$  \hspace{1cm}  \text{(35)}

In the sequel, $P - MN^{-1}MT$ will be referred to as the Schur complement of $N$.

Proof: See [5]. ■

3 Main Result

A synthesis theorem can be presented on the LMI-based solution of the problem now:

Theorem 5 A 1 DOF static state feedback plus integral controller $K \in \mathbb{R}^{m \times n_s}$ exists for the continuous-time $H_\infty$ MMP if and only if there is a matrix

$$X_{cl} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} > 0$$  \hspace{1cm}  \text{(36)}

such that,

$$X_3 \begin{pmatrix} I_m & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} -DT \\ H^T \end{pmatrix} < 0$$  \hspace{1cm}  \text{(37)}

$$N_c \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} > 0$$  \hspace{1cm}  \text{(38)}$$

where $N_c$ is a full rank matrix with

$$ImN_c = Ker\begin{bmatrix} B^T & -D^T & 0_{m \times n_m} & -D^T \end{bmatrix}. \hspace{1cm}  \text{(39)}$$

Proof: From The Bounded Real Lemma, $K \in \mathbb{R}^{m \times n_s}$ is a 1 DOF static state feedback controller in Figure 2 if and only if the LMI

$$\begin{bmatrix} A_{cl}^T & X_{cl} & X_{cl} & C_{cl}^T \\ B_{cl}^T & X_{cl} & X_{cl} & D_{cl} \end{bmatrix} < 0$$  \hspace{1cm}  \text{(40)}$$

holds for some $X_{cl} > 0$ in $\mathbb{R}^{(n_1 + n_m + m) \times (n_1 + n_m + m)}$. Using the expressions $A_{cl}$, $B_{cl}$, $C_{cl}$ and $D_{cl}$ in (16), (17), (18) and (19), this LMI can also be written as:

$$H_{X_{cl}} + P_{X_{cl}}^TQ + Q^T K^T P_{X_{cl}} < 0 \hspace{1cm} \text{(41)}$$

where

$$H_{X_{cl}} = \begin{bmatrix} A_{cl}^T & X_{cl} & X_{cl}B_{cl} & C_{cl}^T \\ B_{cl}^T & X_{cl} & X_{cl} & D_{cl} \end{bmatrix}$$  \hspace{1cm}  \text{(42)}$$

$$Q = \begin{bmatrix} C_{2} & 0_{n_s \times m} & 0_{n_s \times m} \\ 0_{n_s \times m} & -\gamma I_m & -\gamma I_m \end{bmatrix}$$  \hspace{1cm}  \text{(43)}$$

$$P_{X_{cl}} = \begin{bmatrix} B_{cl}^T & X_{cl} & 0_{m \times D_{cl}} \end{bmatrix}.$$  \hspace{1cm}  \text{(44)}$$

I can use Lemma 3 to eliminate the matrix $K$ in the LMI (41). Therefore, the LMI (41) holds for some $K$ if and only if

$$N_{P_{X_{cl}}}^T H_{X_{cl}} N_{P_{X_{cl}}} < 0 \quad \text{and} \quad N_{Q}^T H_{X_{cl}} N_{Q} < 0 \hspace{1cm} \text{(45)}$$

where

$$ImN_{P_{X_{cl}}} = KerP_{X_{cl}}$$  \hspace{1cm}  \text{(46)}$$

$$ImN_{Q} = KerQ$$  \hspace{1cm}  \text{(47)}$$

$$X_{cl} > 0.$$  \hspace{1cm}  \text{(48)}$$
Then, the first inequality in (45) can be rewritten as
\[ N_P^T T_{X_{cl}} N_P \] where the matrix \( N_P \) denotes any basis of \( \text{Ker} P \) and
\[ P = \begin{bmatrix} B_2^T & 0_m & D_2^T \end{bmatrix}. \] (49)

I can take as
\[ P_{X_{cl}} = P \begin{bmatrix} X_{cl} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} \] (50)

hence
\[ N_{P_{X_{cl}}} = \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} N_P. \] (51)

Consequently,
\[ N_{P_{X_{cl}}}^T H_{X_{cl}} N_{P_{X_{cl}}} < 0 \] (52)
is equivalent to
\[ N_P^T \left\{ \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} H_{X_{cl}} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} \right\} \]
\[ . N_P = N_P^T T_{X_{cl}} N_P < 0 \] (53)

where
\[ T_{X_{cl}} = \begin{bmatrix} A X_{cl}^{-1} + X_{cl}^{-1} A^T & B_1 & X_{cl}^{-1} C_1^T \\ B_1^T & -\gamma I_m & D_1^T \\ C_1 X_{cl}^{-1} & D_1 & -\gamma I_m \end{bmatrix}. \] (54)

Meanwhile, from (49) follows that bases of \( \text{Ker} P \) are
\[ N_P = \begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix} \] (55)

where
\[ N_c = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \] (56)
is any basis of the null space of \( \begin{bmatrix} B_2^T & D_2^T \end{bmatrix} \). So the condition
\[ N_P^T T_{X_{cl}} N_P < 0 \] (57)
can be reduced to
\[ \begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix}^T \begin{bmatrix} A X_{cl}^{-1} + X_{cl}^{-1} A^T & B_1 \\ B_1^T & -\gamma I_m \\ C_1 X_{cl}^{-1} & D_1 \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix} < 0 \] (58)
or equivalently
\[ \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^T \begin{bmatrix} A X_{cl}^{-1} + X_{cl}^{-1} A^T & X_{cl}^{-1} C_1^T \\ C_1 X_{cl}^{-1} & -\gamma I_m \\ B_1 & D_1^T \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0. \] (59)

Similarly, in (45) the condition
\[ N_Q^T H_{X_{cl}} N_Q < 0 \] (60)
is equivalent to
\[ \begin{bmatrix} N_o & 0 \\ 0 & I_m \end{bmatrix}^T \begin{bmatrix} A T_{X_{cl}} + X_{cl} A & X_{cl} B_1 \\ B_1^T X_{cl} & -\gamma I_m \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I_m \end{bmatrix} < 0 \] (61)

where
\[ \text{Im} N_o = \text{Ker} \begin{bmatrix} C_2 & 0_{n_x \times m} \end{bmatrix}. \] (62)

Hence the matrix \( X_{cl} \) satisfies the LMI (41) if and only if the matrix \( X_{cl} \) satisfies the LMIs (59) and (61). To complete the proof, it suffices to use (8), (9) and (10) into the LMI (61):
\[ \text{Im} N_o = \text{Ker} \begin{bmatrix} C_2 & 0_{n_x \times m} \\ I_m & 0_{n_x \times m} \\ 0_{n_x \times n_m} & 0_{n_x \times m} \end{bmatrix} \]

and
\[ N_o = \begin{bmatrix} 0_{n_x \times m} & 0 \\ I_m & 0 \\ 0 & I_m \\ 0 & 0 \\ I_m & 0 \end{bmatrix}. \] (63)

Therefore, the following inequality can be derived,
\[ \begin{bmatrix} 0_{n_x \times m} & 0 \\ I_m & 0 \\ 0 & I_m \\ 0 & 0 \\ I_m \end{bmatrix}^T \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} X_{cl} \\ 0 \end{bmatrix} < 0 \]
such that,
\[
\begin{bmatrix}
B^T & 0 & 0 & 0 \\
0_{n_x \times n_x} & 0 & 0 & 0 \\
I_m & 0 & 0 & 0 \\
0 & I_m & 0 & 0 \\
0 & 0 & I_m & 0
\end{bmatrix} X_2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
G^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3
\]
\[
+X_2^T (B \ 0_{n_x \times n_m} + \frac{1}{\gamma} X_3 \begin{bmatrix} I & G^T \end{bmatrix} X_3)
\]
\[
+ \frac{1}{\gamma} \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix} < 0
\]
\[
\begin{bmatrix} W_c & 0 \\
0 & I \end{bmatrix}^T \begin{bmatrix} A & B & 0 & 0 \\
-C & 0 & 0 & F \\
0 & 0 & 0 & 0 \\
0 & 0 & G & G^T \end{bmatrix} X_{cl}^{-1} \begin{bmatrix} A & B & 0 & 0 \\
-C & 0 & 0 & F \\
0 & 0 & 0 & 0 \\
0 & 0 & G & G^T \end{bmatrix} X_{cl}^{-1}
\]
\[
W_c \begin{bmatrix} 0 & 0 \\
0 & I \end{bmatrix} < 0
\]
\[
X_{cl}^{-1} \begin{bmatrix} -C^T & 0 \\
0 & H^T \end{bmatrix} \cdot \begin{bmatrix} W_c & 0 \\
0 & I \end{bmatrix} < 0
\]
where \(W_c\) is a full rank matrix with
\[
\text{Im} W_c = \text{Ker} \begin{bmatrix} B^T & 0 & 0_{n_m \times n_m} \end{bmatrix}.
\]

**Proof:** Let us write the LMI (37) for \(D = 0\) and \(J = 0\):
\[
\begin{bmatrix}
G^T & 0 & 0 & 0 \\
0 & G^T & 0 & 0 \\
I_m & 0 & 0 & 0 \\
0 & H & 0 & 0
\end{bmatrix} X_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3
\]
\[
X_3 \begin{bmatrix} 0 & 0 \\
0 & F \end{bmatrix} + X_2^T (B \ 0_{n_x \times n_m})
\]
\[
X_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3
\]
\[
< 0.
\]

When the Schur complement argument is used, above LMI can be reduced following form:
\[
\begin{bmatrix}
B^T & 0 & 0 & 0 \\
0_{n_x \times n_x} & 0 & 0 & 0 \\
I_m & 0 & 0 & 0 \\
0 & I_m & 0 & 0
\end{bmatrix} X_2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X_3
\]

The strictly proper model system case

Since the system is generally strictly proper in the real life, \(D = 0\) is taken. Moreover the model system can generally be chosen as strictly proper, that is \(J = 0\). Therefore (37) and (66) LMI's can be reduced to more simple form:

**Theorem 6** A 1 DOF static state feedback plus integral controller \(K \in \mathbb{R}^{m \times n}\) exists for the continuous-time \(\mathcal{H}_\infty\) MMP if and only if there is a matrix
\[
X_{cl} = \begin{bmatrix}
X_1 & X_2 & X_3
\end{bmatrix} > 0
\]
\[ +X_2^T \begin{pmatrix} B & 0_{n_x \times n_m} \end{pmatrix} + \frac{1}{\gamma}X_3 \begin{pmatrix} I & G^T \end{pmatrix}X_3 \]
\[ + \frac{1}{\gamma} \begin{pmatrix} 0 & 0 \\ 0 & H^T . H \end{pmatrix} < 0. \] (72)

On the other hand, if \( J = 0 \) is written in (66),
\[ \begin{pmatrix} N_c & 0 \\ 0 & I_m \end{pmatrix}^T \begin{pmatrix} \frac{A}{0} & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & G^T \\ 0 & 0 & G . G^T \end{pmatrix} X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} + \begin{pmatrix} W_c & 0 \\ 0 & I \end{pmatrix} < 0. \] (78)

is written. When the Schur complement argument is used, above LMI can be reduced following form:
\[ N_c^T \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} + X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} N_c < 0. \] (73)

From the equation (39)
\[ ImN_c = Ker \left[ \begin{pmatrix} B^T & 0 & 0_{m \times n_m} \end{pmatrix} \right] \] (75)
or
\[ N_c = \begin{pmatrix} W_c & 0 \\ 0 & I \end{pmatrix} \] (76)

are written. That is, if the equation (77) is used, the LMI (69) is obtained:

5 Controller Construction

Although Theorem 5 is about the solvability conditions of the continuous-time \( \mathcal{H}_\infty \) MMP by the 1 DOF static state feedback with integral control, it also provides a controller construction procedure. Moreover, The MATLAB LMI Control Toolbox [11] can be used to solve LMIs. The controller construction procedure can be summarized as follows:

**Step 1:** Find a solution \( X_{cl} > 0 \) to the LMIs (37) and (66) for \( \gamma_{opt} \) which is the minimal of \( \gamma \).

**Step 2:** Obtain a 1 DOF static state feedback control law \( K \in \mathbb{R}^{m \times n_x} \) in the LMI (41).

In the following section, Theorem 6 and the controller construction algorithm will be used to design a controller to achieve model matching.

6 Numerical Example

Consider the second-order unstable system
\[ G(s) = \frac{s + 0.5}{(s - 1)(s + 0.2)}. \]

The model system is taken as
\[ G_m(s) = \frac{1}{s + 1} \].
The state-space equations of $G(s)$ are obtained as

$$
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{\hat{x}}(t) \\
\dot{q}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0.2 & 0.8 & 1 & 0 \\
-0.5 & -1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\hat{x}(t) \\
q(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
w(t) + 
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} u(t) 
$$

(84)

When I search for a controller, $\gamma_{opt}$, the matrix $X_{cl}$ and the 1 DOF static state feedback controller are obtained as follows:

$$
\gamma_{opt} = 1.144
$$

The state-space equations of $G_m(s)$ are obtained as

$$
\begin{align*}
\dot{q}(t) &= -q(t) + w(t) \quad (81) \\
y_m(t) &= q(t). \quad (82)
\end{align*}
$$

The matrix $F$ is Hurwitz. Since the matrix pair

$$
\begin{bmatrix}
A & B \\
-C & -D
\end{bmatrix}, \begin{bmatrix}
B \\
-D
\end{bmatrix}
$$

is controllable, it is stabilizable. Therefore because of Lemma 1, there is a solution for the continuous-time $H_\infty$ MMP by a 1 DOF static state feedback with integral control. The state-space equations of $P(s)$ in Figure 2 can be given as

$$
\begin{align*}
\dot{x}_1(t) &= 0.1 x_1(t) + 0.2 x_2(t) \\
\dot{x}_2(t) &= 0.8 x_2(t) \\
\dot{q}(t) &= -0.5 \dot{x}_1(t) + 0.8 \dot{x}_2(t) \\
\dot{\hat{x}}(t) &= 0.2 x_1(t) + 0.8 x_2(t) \\
\dot{q}(t) &= -0.5 \dot{x}_1(t) + 0.8 \dot{x}_2(t)
\end{align*}
$$

(85)

$$
\begin{align*}
\dot{z}(t) &= [-0.5 & -1 & 0 & 1] \begin{bmatrix}
x_1(t) \\
x_2(t) \\
\hat{x}(t) \\
q(t)
\end{bmatrix} \\
y(t) &= [1 & 0 & 0 & 0] \begin{bmatrix}
x_1(t) \\
x_2(t) \\
\hat{x}(t) \\
q(t)
\end{bmatrix}
\end{align*}
$$

(86)

Figure 2 can be given as

$$
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{\hat{x}}(t) \\
\dot{q}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0.2 & 0.8 & 1 & 0 \\
-0.5 & -1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\hat{x}(t) \\
q(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
w(t) + 
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} u(t) 
$$

(79)

$$
y_s(t) = [0.5 & 1] \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} \quad (80)
$$

$$
X_{cl} = \begin{bmatrix}
0.5123 & 0.1992 & -0.1094 \\
0.1992 & 0.5126 & -0.1407 \\
-0.1094 & -0.1407 & 0.5294 \\
-0.1501 & -0.1587 & -0.1310
\end{bmatrix} > 0
$$

$$
K = [-1.8655 & -4.1419]
$$

Figure 3. The impulse responses of $G(s) : ..., G_m(s) : ---$ and $T(s) : --.$

Figure 4. The step responses of $G_m(s) : ---$, $T(s) : --.$ and the error function.
Figure 5. The Bode diagrams of \( G(s) : ... \), \( G_m(s) : \ldots \) and \( T(s) : \ldots \)

\( T(s) \) is the closed-loop transfer matrix, i.e. \( G(s) \) with a 1 DOF static state feedback plus integral controller as it is seen in Figure 1. Figure 3 and Figure 4 illustrate the impulse responses and the unit step responses of \( G(s) \), \( G_m(s) \) and \( T(s) \). In Figure 5, the Bode diagrams of \( G(s) \), \( G_m(s) \) and \( T(s) \) are shown. They are matched over \( \gamma_{opt} \). As the figures indicate, the controlled system follows the dynamics of the target system.

7 Conclusions

In this paper, the continuous-time \( H_\infty \) model matching problem by the one degree of freedom static state feedback with an integral controller is investigated. In the previous studies, the \( H_\infty \) model matching problem was not solved by one degree of freedom static state feedback plus integral control which makes zero to the steady-state error.

State feedback control with the integral block is well known, [12]. But in this approach there is no zero assignment. System zeros affect the response of a system a little also. The model matching approach contains poles and zeros assignments. Moreover lots of control problem (The disturbance rejection, robust stability etc...) can be solved by using the LMI theory, [7]. In these problems, the solutions are LMIs. If the disturbance rejection and the model matching problem are wanted to solve simultaneously, the matrix \( X > 0 \) must be found out for all LMI conditions. Therefore it is important to find the LMI conditions of solution of the continuous-time \( H_\infty \) model matching problem by the one degree of freedom static state feedback with an integral controller.

Before the problem is not solved, a block diagram in Figure 1 which is reduced the problem to \( H_\infty \) optimal control problem is proposed and then a synthesis theorem is found out. According to the numerical example, the model matching is really done and the steady-state error is zero. However, the model matching performance can be improved, if two LMIs in Theorem 5 have to be simplified in future.

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References:


