The Optimal Solution of a Production Model with Shortages and Raw Materials

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Abstract: - The purpose of this study is to incorporate the cost of raw material into the classical production model with planned shortages. A mathematical model is developed to account for the cost of producing and holding the finished product, the administrative and backorder costs of shortages, as well as the cost of acquiring and holding the raw materials used in the production process. Explicit expressions for the optimal production and planned shortage quantities are derived from the mathematical model. Moreover, a proof of the uniqueness of the optimal solution is provided. Numerical examples are presented to analyze the sensitivity of the optimal solution relative to changes in the parameters of the model.

Key-Words: - Economic production quantity, planned shortages, cost of raw material

1 Introduction

The economic production quantity (EPQ) model examines the inventory problem of determining the optimal quantity to be produced in order to meet the demand for a certain item. This classical model, which dates back to the early 20th century [1], imposes several simplifying assumptions. Over the past hundred years, the classical EPQ model has been extensively researched by relaxing the imposed assumptions and incorporating real life factors into this model. Factors considered include: shortages, time value of money, credit facilities, deteriorating items, quality, and raw material used in the production process.

The classical EPQ model considers the case where the demand for a certain item is met by producing the item. The model assumes that the production rate α and the demand rate β are constant and known with $\alpha > \beta$. The cost components are the setup cost K, the holding cost per item produced per unit time h_p, and the production cost per unit C_p. For a given production quantity Y, the total cost per unit time function is

$$TCU(Y) = C_p\beta + \frac{C_0\beta}{Y} + \frac{h_p}{2}\left(1 - \frac{\beta}{\alpha}\right)Y. \qquad (1)$$

The optimal quantity to be produced Y^*, the minimizer of the TCU, is given by

$$Y^* = \sqrt{\frac{2C_0\beta}{h_p(1-\beta/\alpha)}}. \qquad (2)$$

One of the assumptions of the classical EPQ model is that shortages are not allowed. However, an extension of the model that allows for planned shortages can be found in most books on inventory models; for instance, see [2].

The various costs associated with raw material used to produce the finished items are not reflected in the classical model. Recently, considerable research work has been dedicated to incorporate the costs of raw material into the EPQ model. Salameh and El-Kassar incorporated the cost of raw material used in the production process into the classical EPQ model [3]. Several papers extended Salameh and El-Kassar model in various directions. For instance, a number of papers studied the effect of quality of raw material on the EPQ model [4], [5],

[6], [7] and [8]. In another direction, the effect of raw material on the EPQ model was considered in the context of supply chain [9], [10], [11] and [12].

The purpose of this paper is to extend the classical EPQ model to incorporate the costs of raw material into the EPQ model with planned shortages. For simplicity, and without loss of generality, we consider the case where a single type of raw material is used in the production process and each unit of the finished product requires one single unit of a raw material. The rest of this paper is structured as follows. The mathematical model is accounting for the cost of producing and holding the finished product, the administrative and backorder costs of shortages, as well as the cost of acquiring and holding the raw materials used in the production process is developed in section 2. Explicit expressions for the optimal production and planned shortage quantities are derived in section 3. A proof of the uniqueness of the optimal solution is also provided in section 3. Numerical examples that analyse the sensitivity of the optimal solution relative to changes in the parameters of the model are presented in section 4. The paper is concluded in section 5.

2 The Mathematical Model

The following notation is used throughout the rest of this paper:

Y	Order size of raw material
S	Size of planned shortage
Z	maximum inventory level
α	Production rate
β	Demand rate
T_p	Length of the production period
T	Length of the inventory cycle
T_1	time to fulfil the backorder of size S
T_2	time to build the maximum inventory level Z
T_3	time to deplete the maximum inventory
T_4	time to build a backorder of size S
C_r	Cost per item raw material
C_p	production cost
C_b	Administrative cost per item short
C_s	Cost per item short per unit time
C_0	ordering cost of raw materials
C_1	setup cost for production
i	inventory holding cost rate
h_r	holding cost of raw materials
h_p	holding cost due to production

The raw materials acquired from a supplier are processed into a finished product at a production rate α. Let Y be the order size of the raw material, an unknown to be determined and let C_r be the cost per unit of raw material. The raw materials are stored and processed at a rate α until it is depleted at the end of the production period. The length of the production period is

$$T_p = Y/\alpha. \quad (3)$$

The inventory level for raw materials is shown in figure 1. During the production period, the finished product is produced at a rate α and consumed at the demand rate β.

To determine the optimal production quantity Y^* and the optimal shortage quantity S^*, we first calculate the total cost per cycle function and then the total cost per unit time function. The cost components per inventory cycle consists of:

i- ordering, purchasing and holding costs of raw materials;
ii- setup cost of production;
iii- production and holding costs of finished product;
iv- shortage and backorder costs.

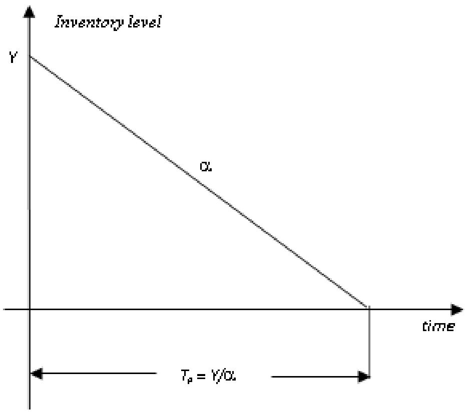

Fig. 1 : Inventory Level of Raw Material

At the start of the production period and until time T_1, the excess amount of the finished product is used to fulfil the S units of backorders at a rate of $\alpha-\beta$. Hence,

$$T_1 = S/(\alpha-\beta). \quad (4)$$

After such time and until the end of the production period, the excess amount of the finished product is used to accumulate inventory of the finished product at a rate of $\alpha-\beta$. This occurs during a time period of T_2, where $T_p = T_1+T_2$. Using (4), we have

$$T_2 = T_p - T_1 = Y/\alpha - S/(\alpha-\beta). \quad (5)$$

At the end of this period, a maximum inventory level Z is reached. Then,

$$Z = T_2(\alpha-\beta) = \left(\frac{Y}{\alpha} - \frac{S}{\alpha-\beta}\right).(\alpha-\beta) \qquad (6)$$
$$= Y(1-\beta/\alpha) - S.$$

This maximum level will be used to meet the demand at a rate β until time T_3 when the inventory level of the finished product reaches zero. Hence,

$$T_3 = \frac{Z}{\beta} = \frac{Y(1-\beta/\alpha)-S}{\beta} = \frac{Y}{\beta}(1-\beta/\alpha) - \frac{S}{\beta}. \qquad (7)$$

Throughout the remainder of the inventory cycle, a planned shortage of size S is built up at the demand rate β until time T_4, where

$$T_4 = \frac{S}{\beta}. \qquad (8)$$

The inventory level for the finished product is shown in figure 2. Note that the length of the inventory cycle is $T = T_1 + T_2 + T_3 + T_4$; that is,

$$T = \frac{Y}{\beta}. \qquad (9)$$

The ordering cost of raw materials is C_0 and the purchasing cost is C_rY. The raw materials holding cost is the holding cost per unit of material per unit time, h_r, multiply by the average on hand inventory of raw materials times the cycle length. From figure 2, we have

Holding Cost of Raw Material =
$$h_r\frac{1}{2}Y\frac{T_p}{T}T = h_r\frac{Y^2}{2\alpha}. \qquad (10)$$

Note that, the holding cost per unit of material per unit time, h_r, is the product of the inventory holding cost rate i and the unit purchasing cost C_r. That is,

$$h_r = iC_r. \qquad (11)$$

The cost of producing the Y units of the finished product is the sum of the setup is C_1 and the production cost is C_pY. The holding cost per unit of the finished product is the sum of h_r and h_p, where $h_p = iC_p$. This is due to the fact that a single unit of the finished product incur both the cost of production as well as the cost of raw material. Thus the finished product holding cost is the average inventory of on hand finished product times the inventory cycle length times the holding cost per unit of a finished product per unit time. From (5)-(7), we have

Holding Cost of Finished Product

$$= (h_r + h_p).\frac{1}{2}.Z.\frac{(T_2+T_3)}{T}.T \qquad (12)$$
$$= \frac{(h_r+h_p)}{2\beta}\left(Y\left(1-\frac{\beta}{\alpha}\right)-S\right)\left(Y - \frac{S\alpha}{\alpha-\beta}\right).$$

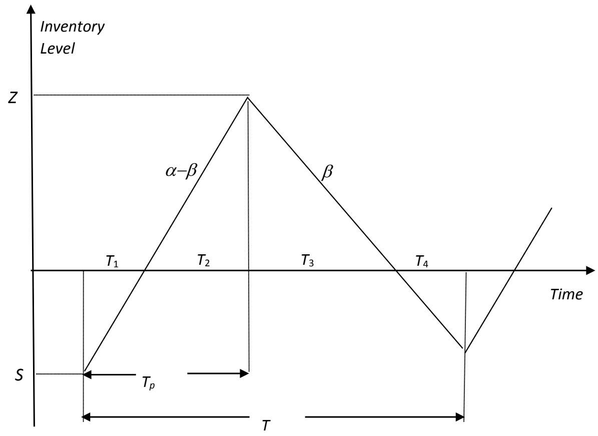

Fig. 2: Finished Product Inventory level

The shortage cost has two components. The first is the time independent administrative cost given by C_bS, while the second is time dependent obtained by multiplying the cost per unit short per unit time C_s by the area below the horizontal axis in figure 3. From (4) and (8), the shortage cost per inventory cycle is

$$\text{Shortage Cost} = C_bS + C_s\frac{1}{2}S\frac{(T_1+T_4)}{T}.T$$
$$= C_bS + C_s\frac{1}{2}S\left(\frac{S}{\alpha-\beta} + \frac{S}{\beta}\right) \qquad (13)$$
$$= C_bS + C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)}.$$

The total inventory cost per cycle function $TC(Y, S)$ is obtained by adding all cost components. Hence,

$$TC(Y,S) = C_0 + C_1 + (C_r + C_p)Y$$
$$+ h_r\frac{Y^2}{2\alpha} + C_bS + C_s\frac{\alpha S^2}{2\beta(\alpha-\beta)} \qquad (14)$$
$$+ (h_r + h_p)\frac{1}{2\beta}\left(Y\left(1-\frac{\beta}{\alpha}\right)-S\right)\left(Y-\frac{S\alpha}{\alpha-\beta}\right).$$

The total inventory cost per cycle function, $TCU(Y, S)$, is obtained by dividing (14) by the cycle length $T = Y/\beta$ so that

$$TCU(Y,S) = (C_r + C_p)\beta + \frac{(C_0 + C_1)\beta}{Y}$$
$$+ h_r \frac{\beta Y}{2\alpha} + \frac{\beta C_b S}{Y} + \frac{C_s S^2}{2(1-\beta/\alpha)Y} \qquad (15)$$
$$+ \frac{h_r + h_p}{2}\left(1 - \frac{\beta}{\alpha} - \frac{S}{Y}\right)\left(Y - \frac{S}{1-\beta/\alpha}\right).$$

3 The Optimal Solution

To obtain the optimal production quantity Y^* and the optimal shortage size S^*, we calculate the first partial derivatives of $TCU(Y, S)$ and set these derivatives equal to zero. Now

$$\frac{\partial TCU}{\partial Y} = -\frac{(C_0 + C_1)\beta}{Y^2} + h_r \frac{\beta}{2\alpha} - \frac{C_s S^2}{2(1-\beta/\alpha)Y^2}$$
$$-\frac{\beta C_b S}{Y^2} + \frac{h_r + h_p}{2}\left(1 - \frac{\beta}{\alpha}\right)\left[1 - \left(\frac{S}{Y(1-\beta/\alpha)}\right)^2\right], \qquad (16)$$

and

$$\frac{\partial TCU}{\partial S} = \frac{\beta C_b}{Y} + \frac{C_s S}{(1-\beta/\alpha)Y}$$
$$+ (h_r + h_p)\left(\frac{S}{Y(1-\beta/\alpha)} - 1\right). \qquad (17)$$

Setting $\partial TCU/\partial S$ equal to zero and multiplying by Y, we have

$$0 = \beta C_b + \frac{C_s S}{(1-\beta/\alpha)}$$
$$+ (h_r + h_p)\left(\frac{S}{(1-\beta/\alpha)} - Y\right). \qquad (18)$$

Solving (18) for Y, yields

$$Y = \frac{\beta C_b}{(h_r + h_p)} + \frac{C_s S}{(1-\beta/\alpha)(h_r + h_p)} + \frac{S}{(1-\beta/\alpha)}. \qquad (19)$$

Setting (16) equal to zero and multiplying by Y^2, we have

$$0 = -(C_0 + C_1)\beta + Y^2 h_r \frac{\beta}{2\alpha} - \beta C_b S - \frac{C_s S^2}{2(1-\beta/\alpha)}$$
$$+ \frac{h_r + h_p}{2}\left(1 - \frac{\beta}{\alpha}\right)\left[Y^2 - \frac{S^2}{(1-\beta/\alpha)^2}\right].$$

Solving for Y, we obtain

$$Y = \sqrt{\frac{2\beta(C_0 + C_1 + C_b S) + \dfrac{S^2(h_r + h_p + C_s)}{(1-\beta/\alpha)}}{h_r \dfrac{\beta}{\alpha} + (h_r + h_p)(1-\beta/\alpha)}}. \qquad (20)$$

Setting the expressions for Y in (19) and (20) equal to each other, squaring and cross multiplying, we get

$$\frac{K\beta^2 C_b{}^2}{(h_r + h_p)^2} + K\left(\frac{C_s}{(h_r + h_p)} + 1\right)^2 \frac{S^2}{(1-\beta/\alpha)^2}$$
$$+ \frac{2K\beta C_b}{(h_r + h_p)} \cdot \left(\frac{C_s}{(h_r + h_p)} + 1\right)\frac{S}{(1-\beta/\alpha)} \qquad (21)$$
$$- 2(1-\beta/\alpha)(C_0 + C_1)\beta - S^2(h_r + h_p)$$
$$- 2(1-\beta/\alpha)\beta C_b S - C_s S^2 = 0$$

where K is the denominator in (20); that is

$$K = (1-\beta/\alpha)h_r \frac{\beta}{\alpha} + (h_r + h_p)(1-\beta/\alpha)^2. \qquad (22)$$

Rearranging the terms of (21), the following quadratic equation is obtained

$$\left[K\left(\frac{C_s}{(h_r + h_p)} + 1\right)^2 \frac{1}{(1-\beta/\alpha)^2} - C_s - (h_r + h_p)\right]S^2$$
$$+ \left[\frac{2K\beta C_b}{(h_r + h_p)} \cdot \left(\frac{C_s}{(h_r + h_p)} + 1\right)\frac{1}{(1-\beta/\alpha)} - 2(1-\beta/\alpha)\beta C_b\right]S$$
$$+ \left(\frac{K\beta^2 C_b{}^2}{(h_r + h_p)^2} - 2(1-\beta/\alpha)(C_0 + C_1)\beta\right) = 0 \qquad (23)$$

Define

$$A = K\left(\frac{C_s}{(h_r + h_p)} + 1\right)^2 \frac{1}{(1-\beta/\alpha)^2} - C_s - (h_r + h_p)$$
$$B = \frac{2K\beta C_b}{(h_r + h_p)} \cdot \left(\frac{C_s}{(h_r + h_p)} + 1\right)\frac{1}{(1-\beta/\alpha)} - 2(1-\beta/\alpha)\beta C_b$$
$$C = \frac{K\beta^2 C_b{}^2}{(h_r + h_p)^2} - 2(1-\beta/\alpha)(C_0 + C_1)\beta. \qquad (24)$$

Then the optimal shortage size is

$$S^* = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \qquad (25)$$

The optimal production quantity is obtained by substituting the value of S^* in (19) so that

$$Y^* = \frac{\beta C_b}{(h_r + h_p)} + \frac{C_s S^*}{(1-\beta/\alpha)(h_r + h_p)} + \frac{S^*}{(1-\beta/\alpha)}. \qquad (26)$$

To demonstrate the uniqueness of the optimal solution, we first calculate the Jacobian matrix with entries equal to the following second partial derivatives:

$$\frac{\partial^2 TCU}{\partial Y^2} = \frac{2\beta(C_0 + C_1 + C_b S)}{Y^3} + \frac{(C_s + h_r + h_p)S^2}{Y^3(1-\beta/\alpha)}, \tag{27}$$

and

$$\frac{\partial^2 TCU}{\partial S^2} = \frac{C_s + h_r + h_p}{(1-\beta/\alpha)Y}. \tag{28}$$

$$\frac{\partial^2 TCU}{\partial Y \partial S} = \frac{\partial^2 TCU}{\partial S \partial Y} = -\frac{(1-\beta/\alpha)\beta C_b + C_s S + (h_r + h_p)S}{(1-\beta/\alpha)Y^2}. \tag{29}$$

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial^2 TCU}{\partial Y^2} & \frac{\partial^2 TCU}{\partial Y \partial S} \\ \frac{\partial^2 TCU}{\partial S \partial Y} & \frac{\partial^2 TCU}{\partial S^2} \end{pmatrix}.$$

The determinant of the Jacobian matrix is always positive since

$$|J| = \frac{2(C_0 + C_1)\beta(C_s + h_r + h_p)}{(1-\beta/\alpha)Y^4} + \frac{(1-\beta/\alpha)\beta^2 C_b^{\,2}}{(1-\beta/\alpha)Y^4}.$$

Since $|J|$, $\frac{\partial^2(TCU(Y,S))}{\partial Y^2}$ and $\frac{\partial^2(TCU(Y,S))}{\partial S^2}$ are all positive, we have that the optimal solution (Y^*,S^*) given in (25) and (26) is the unique minimizer of the $TCU(Y, S)$ function.

Note that the discriminant $B^2 - 4AC$ under the radical in (24) gives a condition on the administrative cost. When simplified, the discriminant is given by

$$\frac{1}{\alpha^2(h_r + h_p)^2}\Big[4\beta(C_s(h_r\alpha + h_p(\alpha-\beta)) + (h_r(h_r + h_p)\beta)(2\alpha(C_s + h_r + h_p)(C_0 + C_1) . + C_b^2\beta(-\alpha + \beta))\Big]$$

Since the first factor in the numerator and the denominator are both positive, the condition on C_b for an optimal solution to exist is

$$C_b < \sqrt{\frac{2(C_s + h_r + h_p)(C_0 + C_1)}{\beta(1-\beta/\alpha)}}. \tag{29}$$

4 Numerical Examples

The daily demand and production rates for a certain item are β = 100 α = 300. The ordering cost of the raw materials used in production is C_0 = \$100 with a unit purchasing cost C_r = \$5. The production and setup cost are C_p = \$10 and C_1 = \$1,250. The holding cost rate is 2% per day so that h_r = 5(0.02) = \$0.1 per item per day, and h_p = 10(0.2) = \$0.2 per item per day. The shortage cost is C_s = \$0.1 per unit per day and the administrative shortage cost is C_b = \$0.

Then K = 0.1556, A = 0.2222, B = 0, and C = −180, so that the optimal solution is to produce Y^* = 1800 units during each production run and to plan for S^* = 900 units of shortages. The corresponding daily total cost of $TCU[Y^*, S^*]$ = \$1,650. The optimal policy calls for an inventory cycle length quantity of T^* = 18 days, production period of T_p^* = 6 days, and a maximum inventory level of Z^* = 300 units.

To study the effects of changes in the parameters of the problem on the optimal solution, the above example was used with varying the values of C_s, and C_1. First, we kept C_b = 0. The results are shown in Table 1.

Table 1: Effects of Changes in C_s and C_1

	$C1$	S	Y	TCU
Cb = 0.				
Cs = 0.	0	516.4	774.6	\$1,525.82
	200	894.4	1341.6	\$1,544.72
	800	1549.2	2323.8	\$1,577.46
	1400	2000.0	3000.0	\$1,600.00
Cs = 0.1				
	0	244.9	489.9	\$1,540.82
	200	424.3	848.5	\$1,570.71
	800	734.8	1469.7	\$1,622.47
	2000	1122.5	2245.0	\$1,687.08
Cs = 0.5				
	0	88.9	355.4	\$1,556.27
	600	235.1	940.3	\$1,648.88
	1000	294.7	1178.8	\$1,686.64
	2000	407.2	1628.7	\$1,757.88

The results of Table 1 indicate that when the administrative shortage is \$0, the size of the number of planned shortage relative to the lot size is high when the time dependent shortage cost is low and that relative size decreases as the shortage cost increases. We also note that the total cost per unit time is not very sensitive to change in the time dependent shortage cost and the ordering cost of raw material.

4 Conclusion

The model presented in this paper accounts for the costs due to raw material on the classical economic production model with planned shortages. The mathematical models was derived and explicit expressions for the optimal production and shortage quantities were obtained. A proof of the uniqueness of the optimal solution was presented. Numerical examples were given to illustrate the model and to examine the sensitivity of the optimal solution relative to changes in the parameters of the model.

For future work, we suggest incorporating the effects of quality of the raw materials in this model. Also, we suggest studying the effects of quality of the finished product where on these models by considering reworking and scraping the imperfect quality finished items. In other direction, we suggest considering the model presented in the paper in the supply chain context.

References:

[1] F.W. Harris, What quantity to make at once. In: The Library of Factory Management. Operation and Costs, *The Factory Management Series*, Vol. 5, pp. 47–52, A.W. Shaw Co., Chicago, 1913.

[2] L. A. Johnson, D. C. Montgomery, *Operations research in production planning, scheduling, and inventory control*, Vol. 6, New York: Wiley, 1974.

[3] M. K. Salameh, A. N. El-Kassar, Accounting for the Holding Cost of Raw Material in the Production Model. *In Proceeding of BIMA Inaugural Conference*, March 2007, Sharjah, pp. 72-81.

[4] N. Yassine, Joint Probability Distribution and the Minimum of a Set of Normalized Random Variables, *Procedia-Social and Behavioral Sciences*, 2016, Vol. 230, pp. 235-239.

[5] A. N. El-Kassar, M. K. Salameh, M. Bitar, EPQ model with imperfect quality raw material, *Mathematica Balkanica*, 2012, Vol. 26, pp. 123-132.

[6] A. N. El-Kassar, M. K. Salameh, M. Bitar, Effects of time value of money on the EPQ Model with the imperfect quality items of Raw material, *Proceedings of Academy of Information and Management Sciences*, New Orleans, 2012, Vol. 16, No. 1, pp. 11-18.

[7] T. Mikdashi, A.N. El-Kassar, A. Joudi, Effects of Probabilistic Imperfect Quality Items of Raw Material on the Optimal Production Lot Size, *Journal of the Academy of Business & Economics*, 2012, Vol. 12, No. 3, pp. 75-84.

[8] M. Khan, M. Y. Jaber, A. L. Guiffrida, S. Zolfaghari, A review of the extensions of a modified EOQ model for imperfect quality items, 2011, Vol. 132, No. 1, pp. 1-12.

[9] S. Pal, G. S. Mahapatra, G. P. Samanta, A Three-Layer Supply Chain EPQ Model for Price-and Stock-Dependent Stochastic Demand with Imperfect Item Under Rework, *Journal of Uncertainty Analysis and Applications*, 2016, Vol. 4, No. 1, 10.

[10] S. Pal, G. S. Mahapatra, A manufacturing-oriented supply chain model for imperfect quality with inspection errors, stochastic demand under rework and shortages, *Computers & Industrial Engineering*, 2017, 106, 299-314.

[11] M. Khan, M. Y. Jaber, Optimal inventory cycle in a two-stage supply chain incorporating imperfect items from suppliers, 2011, *International Journal of Operational Research*, Vol. 10, No. 4, pp. 442-457.

[12] S. S. Sana, A production-inventory model of imperfect quality products in a three-layer supply chain, *Decision Support Systems*, 2011, Vol. 50, No. 2, pp. 539-547.